THE APPLICATION OF RANDOM VORTEX METHOD IN THE ANALYSIS OF HEAT CONDUCTION IN A RECTANGULAR SLAB

Olusoji Ofi
*Ademola A. Dare
Department of Mechanical Engineering
University of Ibadan, Ibadan, Nigeria.
*E-mail: ademola.dare@mail.ui.edu.ng,
ademola_dare@yahoo.com

ABSTRACT

Thermal conduction problems have traditionally been solved using analytical or numerical tools. Deterministic numerical tools such as Finite Element and Finite Difference Methods have been common. However these methods require many mathematical and computational skills. In the light of this, random vortex method which is probabilistic in nature was applied to study the heat conduction in a rectangular slab. Two cases were considered: one with the slab boundary held at constant temperature and the other with part of the boundary isothermal and part adiabatic. The generated results were then validated using Taylor's series approximation test. From the results, the maximum deviation from the Taylor's series tests was 0.10 for the purely isothermal boundary slab and 0.07 for slab with partly isothermal and partly adiabatic boundary. The results therefore justify the use of random vortex method for modeling heat conduction.
Keywords: Heat Conduction, Simulation, Slab, Probability method, Vortex method

INTRODUCTION

Thermal conduction problems have in time past been solved using analytical or numerical tools. Such tools are well documented by Incroprera and deWitt (1990) as well as Welty, Wicks, Wilson and Rorrer (1989). Finite Element Method and Finite Difference Method have been domineering as reflected in the works of Jing, Antonios, Martin and Joanna (2002) and Tasarkuyu and. Akinoglu (2004). Probability methods have also been attempted. For instance Haji-Sheikh and Sparrow (1967) used the probability method to solve for temperature distribution in a rectangular slab. This approach was further extended by Ogundare (1990) to solve for temperature distribution in an arbitrary surface. A more recent probability approach was reported by Grigorin (2000). Leveque and Rezzong (1999) carried out thermal studies of a superconductivity current limiter using Monte Carlo method. Probability method application to Bio-heat transfer was reported by Zhong-Shan and Jing (2002).

Vortex method was first introduced as a numerical tool for fluid flow studies by Chorin in 1973. Since then, it has grown formidably in its application to fluid flow. Many versions of it reported by Liu et al. (2005) include random vortex, core expansion, and particle strength exchange, and diffusion velocity techniques. Many application cases attempted include the study of hydrodynamic boundary layer in fluids (Lewis, 1991) and forced convection over flat plate (Shen and Lu, 1985). A diffusion velocity version of the technique was recently applied to study natural diffusion over vertical plate (Ogami, 1999).

By carefully studying the approach adopted in the use of vortex method for solving fluid flow problems, it is observed that it can be adopted to solve thermal conduction problems. This is attempted in this work.

Description of the Theory of Vortex Method and Application to Thermal Conduction

The vorticity equation for a 2-D fluid flow is given as

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = v\frac{\partial^2 \omega}{\partial y^2} \qquad \text{--(1)}$$

  $x =$ displacement in horizontal direction(m)
  $y =$ displacement in vertical direction (m)
  $v =$ kinematic viscosity (m^2/s)
  $\omega =$ vorticity (s^{-1})
  $u =$ horizontal component of velocity (m/s)
  $v =$ vertical component of velocity (m/s)
  $t =$ time (s)
 and $r = \sqrt{x^2 + y^2}$ = radial shift (m)

When only diffusive terms are present, then the vorticity equation trims down to

$$\frac{\partial \omega}{\partial t} = v\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \qquad \text{--(2)}$$

The solution for this equation is given by Batchelor(1967) as

$$\omega(r,t) = \frac{\Gamma}{4\pi \upsilon t} e^{\left(\frac{-r^2}{4\upsilon t}\right)} \qquad \text{--(3)}$$

where $\Gamma =$ Circulation (m^2/s)

Eqn. (3) resembles the normal distribution probability density function.

It is on this premise that a random walk model was proposed by Chorin (1973) for simulating boundary layer flow. Equation 3 suggests that a velocity vortex element i will undergo displacement during time t given as

$$\theta_i = 2\pi Q_i \qquad \text{--(4)}$$

$$r_i = 4\upsilon t \ln\left(\frac{1}{P_i}\right)^{1/2} \qquad \text{---(5)}$$

Then its new position x_i' y_i' will be

$x_i' = x_i + r_i \cos\theta_i$ --(7)
$y_i' = y_i + r_i \sin\theta_i$ --(8)

Q and P are random numbers with values between 0 and 1.

If therefore elements with known vorticity are created at the wall, they can be diffused into space using eqns. (7) and (8). The vortex strength for each element i at the wall is given as $\Gamma = u_i \Delta s$ where u_i is the slip velocity and Δs is element size. A typical flow chart for fluid flow is shown in Fig. 1.
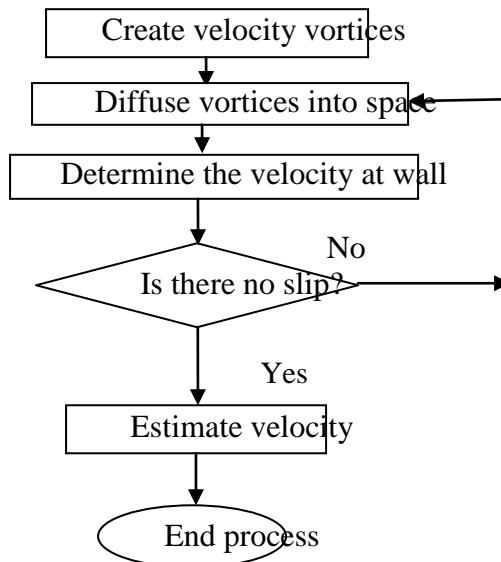
Fig. 1: Algorithm for Vortex Simulation of Diffusion Fluid Flow

The governing equation for the 2-dimensional unsteady state thermal conduction without heat generation is given as:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

α = thermal diffusivity (m^2/s)

T= temperature (K)

This equation is similar to the vorticity equation The diffusion is thus simulated by replacing kinematic viscosity, ν with thermal diffusivity term, α. The temperature vortex strength is given as $\psi = -\alpha \Delta t \left(\frac{\partial T}{\partial y} \right)_0$; y is in the direction normal to the wall and Δt is the time step. A steady value is obtained when ' no slip' of temperature at the wall has been achieved. A comparative flow chart is shown in Fig. 2.
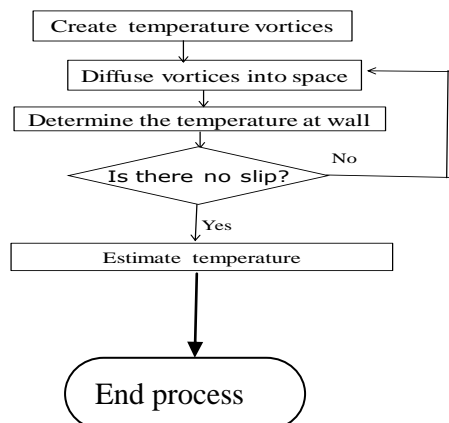

Fig. 2: Algorithm for Vortex Simulation of Temperature Distribution

Simulation Results for Steady State Thermal Conduction Problems

Two hypothetical cases were considered in this instance. First a slab with the boundary held at constant temperature and with no heat generation was considered. Then a slab with part of the boundary at constant temperature and part of it adiabatic was considered. The computations were carried out for two dimensional steady state conduction cases. The rectangular slabs with appropriate dimensions are shown in Figures 3 and 4.
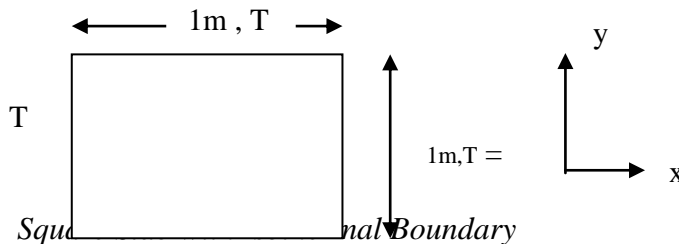

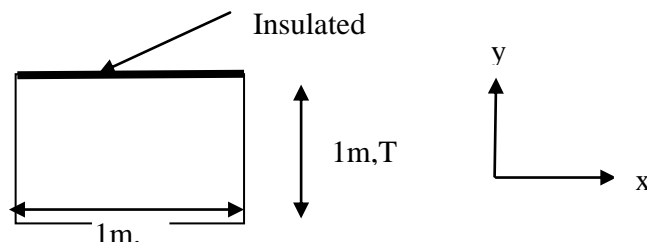**Fig. 3:** *Square ... Boundary*

Fig. 4 Slab with Upper Boundary Insulated

Case a: Rectangular Slab with Constant Boundary Temperature

The basic parameters for simulation

(i) Slab dimension = 1m x 1m (ii) Slab material = mild steel

The temperatures are specified in dimensionless format, $\Theta = T/T_o$, $\Theta = \dfrac{T}{T_0}$,

where T_o is the maximum boundary temperature, T is the temperature of any point in the domain. For the case thus considered, Θ takes the value of 1 at all the boundaries. The slab initial temperature is set to 0 K. The results of the simulation for a 10 x 10 grids are presented on Table 1.

Table 1 Temperature Solution for Slab with Isothermal Case

10	1	1	1	1	1	1	1	1	1	1
9	1	0.96	0.81	0.77	0.76	0.76	0.76	0.81	0.96	1
8	1	0.7	0.43	0.33	0.32	0.32	0.33	0.43	0.7	1
7	1	0.56	0.22	0.1	0.08	0.07	0.1	0.22	0.56	1
6	1	0.53	0.17	0.04	0.01	0.01	0.04	0.17	0.53	1
5	1	0.53	0.17	0.04	0.01	0.01	0.04	0.17	0.53	1
4	1	0.56	0.22	0.1	0.07	0.07	0.1	0.22	0.56	1
3	1	0.7	0.43	0.33	0.31	0.31	0.33	0.43	0.7	1
2	1	0.95	0.8	0.76	0.76	0.76	0.76	0.81	0.95	1
1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10

↑ Grid's Nomenclature (N-S)

Grid's Nomenclature (E-W)

* Boundary values are in bold format. Other values are for grid nodes in the domain.

The finite difference representation of a solution point in a domain is given as $T_i = (T_1 + T_2 + T_3 + T_4)/4$ for square grids. T_1, T_2, T_3 and T_4 are neighbours of T_i. Ten sample points are located at the solution domain and the validity of this relation determined. The sample points are shown shaded in Table 1. They are numbered from top to bottom and left to right as on Table 2.

Table 2 Sampled Solution Points and Deviation Test Results.

Solution point	Coordinate	Sample value	Relation value	Deviation
1	5,8	0.32	0.37	.05
2	3,7	0.22	0.32	.10
3	9,7	0.56	0.59	.03
4	6,6	0.01	0.03	.02
5	3,5	0.17	0.23	.06
6	4,4	0.1	0.17	.07
7	6,3	0.31	0.38	.07
8	8,3	0.43	0.51	.07
9	9,3	0.70	0.73	.03
10	3,2	0.8	0.79	0.01

From the table 2, it could be noticed that the maximum deviation was not more than .10 while the deviation values generally lie below 0.07.

Case b: Slab with Upper Boundary Insulated

The basic parameters remain the same as in Case a. The results also for a 10 x 10 grids (boundary inclusive) are also presented on Table 3. The validation table was also constructed as in Case a. This is presented on Table 4.

Table 3: Temperature Distribution in A Slab with Upper Boundary Insulated

Adiabatic surface

	1	2	3	4	5	6	7	8	9	10	
10	**0.8**	0.5	0.19	0.04	0.01	0.01	0.05	0.19	0.5	0.8	
9	**1**	0.6	0.23	0.05	0.01	0.01	0.05	0.23	0.61	**1**	
8	**1**	0.6	0.23	0.05	0.01	0.01	0.05	0.23	0.6	**1**	
7	**1**	0.59	0.22	0.05	0.01	0.01	0.05	0.23	0.59	**1**	
6	**1**	0.59	0.23	0.05	0.01	0.01	0.05	0.23	0.6	**1**	
5	**1**	0.6	0.24	0.07	0.03	0.03	0.07	0.24	0.6	**1**	
4	**1**	0.64	0.32	0.16	0.12	0.12	0.16	0.32	0.65	**1**	
3	**1**	0.79	0.54	0.42	0.39	0.39	0.42	0.54	0.79	**1**	
2	**1**	0.99	0.86	0.81	0.8	0.8	0.81	0.86	0.99	**1**	
1	**1**	**1.**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	
	1	2	3	4	5	6	7	8	9	10	Grid's Nomenclature (E-W)

Grid's Nomenclature (N-S)

* Boundary values are in bold format while sampled results are shown shaded.

In this case, the maximum deviation is only 0.07. It is however that noticed those sample points with smaller values have larger deviations. In general, the solutions seem comparable with the finite difference method.

Table 4: Validation Table for Slab with Upper Boundary Insulated

Sample identity	Coordinate	Sample value	Relation value	Deviation
1	4,10	0.04	0.08	0.04
2	8,10	0.19	0.25	0.06
3	2,9	0.60	0.58	-0.02
4	7,8	0.05	0.08	0.03
5	3,6	0.23	0.27	0.04
6	6,6	0.01	0.02	0.01
7	9,5	0.24	0.31	0.07
8	5,4	0.12	0.18	0.06
9	2,3	0.79	0.79	0.0
10	4,2	0.81	0.77	-0.04
11	8,2	0.86	0.84	-0.02

Comparative Temperature Profiles for Cases 'a' and 'b'
The temperature profile surface plots for the slab for the two cases are shown in Figs. 5 and 6 respectively while the contour plots for the two cases are shown in Figs. 7 and 8. Expectedly, the profiles compare sharply since the adiabatic surface permits minimal or no heat transfer.
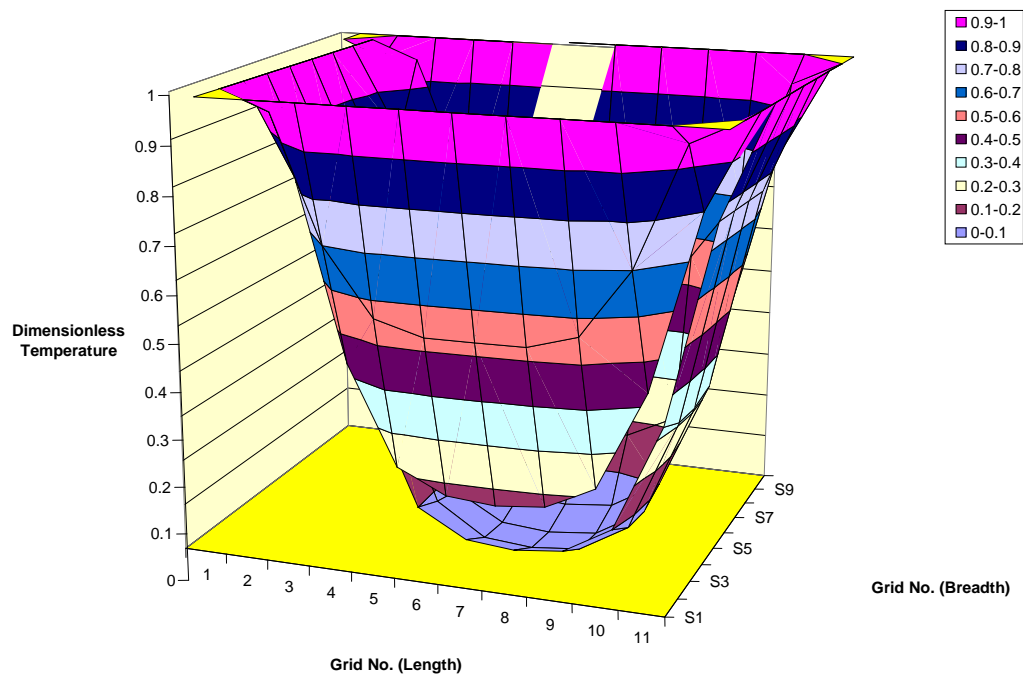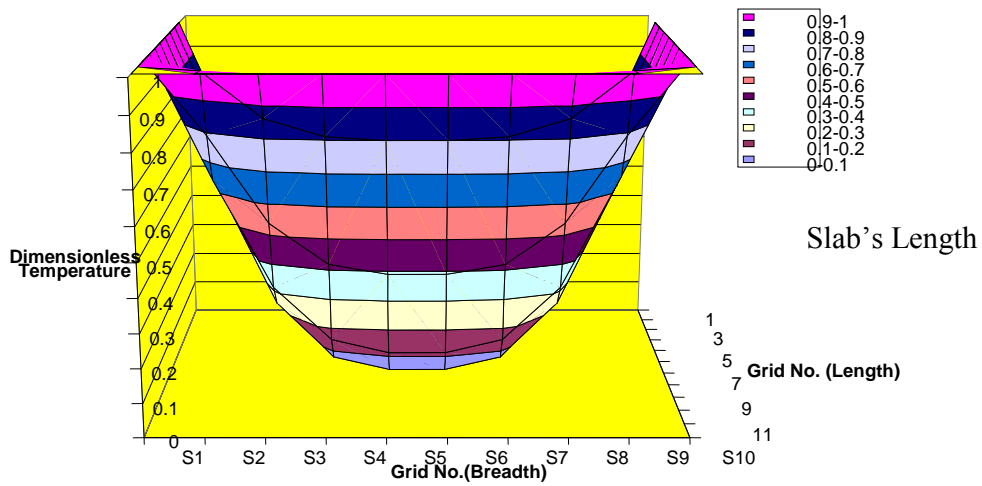

Fig. 5: Temperature Trend in The Isothermal Slab

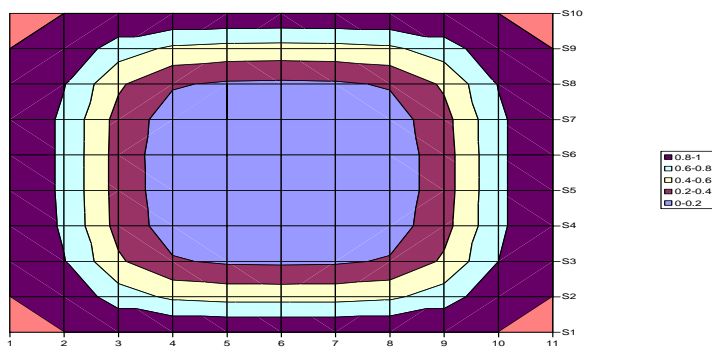**Fig. 6:** Temperature Trend in a Slab with Upper Surface Insulated

Fig. 7: Contour Plot of Temperature in A Slab with Isothermal Boundary
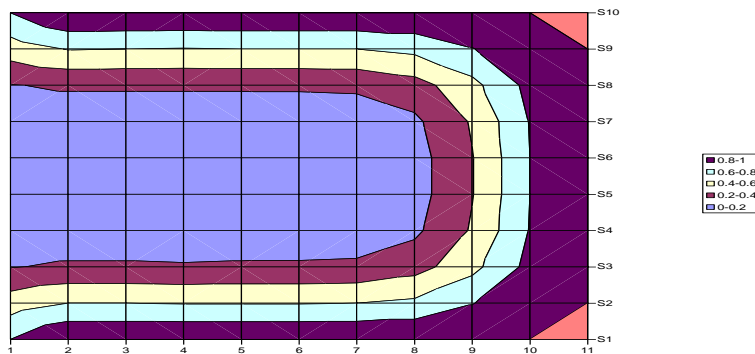

Fig. 8: Contour Plot of Temperature in A Slab with Upper Surface Insulated

CONCLUDING REMARK

Analytical or numerical tools have traditionally been used to solve Thermal conduction problems. Deterministic numerical tools such as Finite Element and Finite Difference Methods have been common. However these methods require many mathematical and computational skills. From this study, vortex method is established as being viable for simulation of thermal conduction problems. In view of this adaptability, it is possible to simulate other engineering phenomena with similar governing equations using vortex method.

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