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Iterative Approximation of Nonlinear Fredholm Integral Equations of the Second Kind by Picard's Method

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ABSTRACT

This study on Iterative Approximation of Nonlinear Fredholm Integral Equations of the Second Kind by Picard's Method considers the application of Picard's iteration scheme for the approximation of an operator equation in Banach space. Using Lipschitz continuity condition and the prescribed auxiliary scalar function, the location of existence of solution for nonlinear integral equation Fredholm type and second kind is obtained. The error estimate provided in the analysis is used to predict the convergence speed of the Picard's scheme. An indication from the error estimate shows that the error will be totally insignificant after eight iterations.

Keywords: *Fredholm integral equations, Picard's iteration, Lipschitz continuity, Fréchet derivative*
MSC 2010: *45B05, 46N40, 47H10*

INTRODUCTION

The existence of solution of a nonlinear equation of Fredholm type

$$x(t) = f(s) + \lambda \int_a^b K(s,t)H\big(x,(t)\big)dt, \quad s \in [a,b] \qquad (1)$$

considered earlier by Ezquerro, and Henandez (2004) based on Picard iteration (Song-bai and Hui-fu, 2001; Rainey, Aghalaya and Ross, 2017)

$$x_{n+1} = G(x_n) = x_n - F(x_n) \qquad (2)$$

was also studied in this work.

The parameter λ in Equation (1) is a real number and the kernel $K(s,t)$ is a continuous function in $[a,b] \times [a,b]$ and $H: C[a,b] \to C[a,b]$ is a differentiable operator in $[a,b]$. In addition, H' is Lipschitz continuous in the

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domain of differentiation. For the purpose of this study, we consider a particular case of (1) where the Kernel is a green function

with equation (1) expressed as

$$G(x)(s) = f(s) + \lambda \int_a^b K(s,t)H\big(x,(t)\big)dt. \qquad (4)$$

Let F be an operator such that

$$F(x) = 0 \qquad (5)$$

where

$$F: D \subseteq X \to X$$

is an operator defined on an open convex domain D of a Banach space X with values in X. It follows from equations (1) and (4) that imply

$$[F(x)](s) = x(s) - G(x)(s)$$

and

$$[F(x)](s) = x(s) - f(s) + \lambda \int_a^b K(s,t)H\big(x,(t)\big)dt, \qquad (6)$$

with $X = C[a,b]$,a space of continuous functions equipped with a max norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|, x \in X.$$

The linear operator associated with (5) is the first derivative in Fréchet sense defined for every $x \in D$ by the formula

$$[F'(x)y](s) = y(s) - \lambda \int_a^b K(s,t)H\big(x,(t)\big)dt, s \in [a,b], y \in X \qquad (7)$$

We shall employ a Picard type scheme (Yang, and Liu, 2014) to equation 5 in order to obtain the result on the existence of solutions of such equations. We will rely on the ideas previously considered by Davies (1962), Ahues (2004), Ezquerro and Henandez (2004) and Guitierrez, Hernandez, and Salanova, (2004) under different techniques and assumptions. In the next section, we present the main result of this work with numerical consideration following afterwards.

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RESULTS AND DISCUSSION

Let

$$M = \max_{[0,1]} \int_a^b |K(s,t)|dt \text{ and} x_0 \in D \text{ such that} \|x_1 - x_0\| \leq \eta,$$

next, we consider the following lemmas based on the auxiliary scalar function

$$f(t) = bt^2 - \eta t + \eta^2 \quad (8)$$

Lemma 1

Assuming that $f(t) = 0$ has at least a positive real solution with r denoting the smaller one, then the following relations hold

(i) $\eta < r$
(ii) $a = \lambda ML < 1, L > 0$
(iii) $b = a\|x_1 - x_0\| < 1$
(iv) $r = \dfrac{\eta}{1-\omega(r)}$, where $\omega(t) = at$.

Proof:

Let $f(r) = 0$, then $br^2 - \eta r + \eta^2 = 0$

$(br - \eta)r = -\eta^2$ and $r = \dfrac{\eta^2}{\eta - br} = \dfrac{\eta}{1-\omega(r)}$. Thus, (iv) is established.

Also, $1 > 1 - \omega(r) > 0$, then $1 < \dfrac{1}{1-\omega(r)} < \dfrac{r}{\eta}$, so that $\eta < r$ and (i) holds

Again, $f(r) = 0, r\eta - \eta^2 = br^2, then\ \eta(r - \eta) = br^2$ and

$r - \eta = ar^2$, then $a = \dfrac{r-\eta}{r^2} < 1\ since\ r - \eta > 0$, this establishes (ii).

As $r > 0, \omega(r) = ar < 1 \Rightarrow a\eta < 1 = a\|x_1 - x_0\| = b < 1$ and (iii) holds.

Let $B(x_0, r) = \{x \in X: \|x_1 - x_0\| < r\}$ and

$\bar{B}(x_0, r) = \{x \in X: \|x_1 - x_0\| \leq r\}$

Then we have the following Lemma.

Lemma 2

If $B(x_0,r) \subseteq D$, then for all $x \in B(x_0,r)$

(i) $\|F'(x) - F'(x_0)\| \le \omega(r)$

(ii) If $x_n, x_{n-1} \in B(x_0,r), then\ \|F(x_n)\| \le a\|x_n - x_{n-1}\|^2$

Proof

(i) $\|F'(x) - F'(x_0)\| \le |\lambda|M\|H'(x) - H'(x_0)\|$

$$\le |\lambda|ML\|x - x_0\|$$

$$\le ar = \omega(r).$$

(ii) To prove (ii), we consider the Taylor formula (Rall, 1969).

$$F(x_n)s = \int_0^1 [F'(x_{n-1} + \rho(x_n - x_{n-1}) - F'(x_{n-1})](x_n - x_{n-1})d\rho$$

$$= -\lambda \int_0^1 \int_a^b K(s,t)[H'(x_n)(t) - H'(x_{n-1})(t)](x_n(t) - x_{n-1}(t))dtd\rho$$

where

$x_n(\rho) = x_{n-1} + \rho(x_n - x_{n-1}), \rho \in [0,1]$. If H' is Lipschitz continuous, then

$$\|F(x_n)\| \le \lambda ML\|x_n - x_{n-1}\|^2 = a\|x_n - x_{n-1}\|^2.$$

Theorem 1

Let *f*(*t*) = 0 with $f(t) = bt^2 - \eta t + \eta^2$ have at least a positive solution and let *r* be the smaller one. If $B(x_0,r) \subseteq D$, then there exists at least a solution x^* of (4) in $\bar{B}(x_0,r)$.

Proof

From Lemma 1, we have that

$$\|x_1 - x_0\| \le \eta < r, \text{hence}, x_1 \in B(x_0,r)$$

then

Therefore,

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By (iv) of Lemma 1,

$$\eta(\omega(r)+1) < r.$$

Thus,

$$x_2 \in B(x_0, r).$$

By induction,

(9)

hence by fixed point theorem,

$$\|x_n - x_0\| \le \left(\sum_{k=0}^{n-1} \omega(r)^k\right) \|x_1 - x_0\|$$

$$< \left(\sum_{k=0}^{\infty} \omega(r)^k\right) \eta = r.$$

Consequently, $x_2 \in B(x_0, r)$ for all $n \ge 0$.

$$\|x_2 - x_0\| \le \omega(r)\eta + \eta$$

$$\|x_{n+1} - x_n\| \le \omega(r)^n \|x_1 - x_0\|$$

$$\le \frac{1-\omega(r)^m}{1-\omega(r)} \omega(r)^n \|x_1 - x_0\|.$$

Next, we show that $\{x_n\}$ s a Cauchy sequence.From (9), Lemma 1 and the fixed-point theorem, we have

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_n - x_{n-1}\|$$

Since $\omega(r) = ar < 1$, the numerator $1 - \omega(r)^m < 1$ and

$$\|x_{n+m} - x_n\| \le \frac{\omega(r)^n}{1-\omega(r)} \|x_1 - x_0\|.$$

Setting $\lim_{m\to\infty} x_m = x^*$, we have

$$\|x_n - x^*\| \le \frac{\omega(r)^n}{1-\omega(r)} \|x_1 - x_0\|.$$

Finally, for $n = 0$,

$$\|x^* - x_0\| \le \frac{\eta}{1-\omega(r)} = r.$$

Therefore, $x^* \in B(x_0, r)$ and $F(x^*) = \lim_{n\to\infty} \|F(x_n)\| = 0$. Hence, $x^*$is a solution of $F(x) = 0$.

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Applications

The above theoretical result is illustrated with the following example. Consider the second kind nonlinear integral equation of Fredholm type

$$x(s) = s + \lambda \int_0^1 K(s,t)x(t)^2 dt, s \in [0,1] \qquad (10)$$

Let $X = C[0,1]$ be a space of continuous functions equipped with the max norm and $K(s,t)$ is the Green function (3). The operator F associated with (10) is given by

$$F(x)(s) = x(s) - s - \lambda \int_0^1 K(s,t)x(t)^2 dt$$

and

$$F^{'}(x)y(s) = y(s) - 2\lambda \int_0^1 K(s,t)x(t)y(t)dt, s \in [0,1], y \in X.$$

For $x_0 = 0, \|F(x_0)\| = \eta = 1$ and $b = a.$

$$\|F^{'}(x) - F'(x_0)\| \leq \frac{|\lambda|}{4}\|x - x_0\|$$

Choosing $a = \frac{|\lambda|}{4}$, then $\omega(r) = \frac{|\lambda|}{4} r$

and the equation $f(t)$ with f in (8) is given by

$$\frac{|\lambda|}{4} t^2 - t + 1 = 0 \qquad (11)$$

By choosing $\lambda = \frac{1}{2}$, then the iteration (2) starting from $x_0(s) = 0$ has a fixed point in the ball $\bar{B}(x_0, 4 - 2\sqrt{2})$.

The first three iterations are

$x_1(s) = s$

$x_2(s) = 1{,}041667s - 0.041667s^4$

The approximate error after three iterations can be obtained from theorem 1 as

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$$\leq \frac{\left(\frac{1}{8}\right)^3 (1.171572)^3}{1 - \left(\frac{1}{8}\right)(1.171572)} = 3.68 \times 10^{-3}.$$

An indication from the error estimate shows that the error will be totally insignificant after eight iterations.

CONCLUSION

The analysis of Picard's method for the solution of nonlinear Fredholm integral equations of the second kind was studied in this study. The existence of the solution based on fixed-point theorem was demonstrated. The analysis included a bound for error in the procedure, which gives information on convergence and computation time of the iterative process.

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