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Numerical Solution of Volterra Integral Equations of the Second Kind Based on Sinc Collocation Method with the Error Function

Eno John
Department of General Studies, Akwa Ibom State Polytechnic, Ikot Osurua
eno.john@akwaibompoly.edu.ng

Promise Asukwo
Department of Statistics, Federal Polytechnic, Ukana
promiseasukwo@gmail.com

Nkem Ogbonna
Department of Mathematics, Michael Okpara University of Agriculture, Umudike
Ogbonna.n42@gmail.com

ABSTRACT

The paper investigates the application of error function for the solution of Volterra integral equations of the second kind. Using Sinc spaces of approximation, a collocation procedure was developed with the error function as a variable transformation function for the conversion of the integral equation into algebraic equations. The approximate solution obtained through the algebraic process is studied alongside the approximate solutions based on hyperbolic tanh function over the same interval to compare their rates of convergence based on their deviation from the exact solution. Numerical examples are given to demonstrate the efficiency of this method.

Keywords and phrases: *Sinc function, error function, trapezoidal rule, collocation method, Volterra integral equations*
2010 Mathematical Subject Classification:*45B05, 65L60*

INTRODUCTION

This paper considered the application of error function for the numerical solution of the Volterra integral equations of the second kind.

Let φ be a variable transformation function, then the definite integral I can be expressed as

$$I = \int_a^b f(x)dx = \int_{-\infty}^{\infty} f(\varphi(t)\varphi'(t)dt \tag{1}$$

after a one-to-one transformation of the original integrand $f(x)$ by the function $\varphi(t)$.

Generally, for a function F(u) defined on (-∞, ∞) with integral

$$I = \int_{-\infty}^{\infty} F(u)du, \tag{2}$$

and a real number $h > 0$, the trapezoidal approximation T_h to I is defined by

$$T_h = h \sum_{j=-\infty}^{\infty} F(jh). \tag{3}$$

Sinc methods are based on the Sinc approximation on the whole real line, expressed as

$$F(u) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(u), \quad u \in \mathbb{R} \tag{4}$$

The Sinc quadrature is derived by integrating both sides of (4), that is

$$\int_{-\infty}^{\infty} F(u)du \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{\infty} S(j,h)(u)du = h \sum_{j=-N}^{N} F(jh) \tag{5}$$

which is a truncated trapezoidal formula with

$$S(j,h)(t) = S\left(\frac{t}{h} - j\right) = \frac{sin\pi(\frac{t}{h} - j)}{\pi(\frac{t}{h} - j)} \tag{6}$$

known as called the Sinc function.

Using (1) and (4), (5), we have

$$I = \int_a^b f(x)dx = \int_{-\infty}^{\infty} f(\varphi(t)\varphi'(t)dt \approx h \sum_{j=-N}^{N} f\big(\varphi(jh)\big)\varphi'(jh). \tag{7}$$

2.0 The Collocation Method

We consider in this section a numerical scheme for the solution of integral equations of the form

$$u(x) = \lambda \int_a^x k(x,t)u(t)dt + g(x),\ a \le x \le b \tag{8}$$

which is called Volterra integral equations of the second kind Wazwaz (2011), John (2019). Here, k and g are given smooth functions and u is the solution to be determined.

Some of the approaches that have so far been employed towards approximating its solution include, Adomian decomposition method and Taylor polynomials method (Maleknejad & Mahmoudi 2003, Wazwaz 2011, Okayama, Matsuo & Sugihara 2011, Maleknejad & Nadaisi 2011, John 2013, Mohsen & El-Gamel 2014, John 2016, Maleknejad & Ostadi 2016, & Maleknejad, Rostami & Kalalagh 2016). The Trapezoidal Nystrom method was employed for the solution of linear Volterra-Fredholm integral equations (John & Ogbonna 2016), and the asymptotic expansion error for the method obtained.

2.1 Collocation Formula for Volterra Integral Equations

Let $u(x) \in M_\alpha(\varphi(D))$ and $u_N(x)$ be the exact and approximate solution of (8), while $u(x_j)$ and u_j are the exact and approximate solutions at a sinc point x_j respectively,
Following Stenger (1993) & Haber (1993),

$$u(x) = u(x_{-N-1})w_a(x) + \sum_{j=-N}^{N} u(x_j)S(j,h)\big(\{\varphi\}^{-1}(x)\big) + u(x_{N+1})w_b(x) \tag{9}$$

will satisfy (8) at sinc points of φ, since it is a linear combination of the sinc functions $S(j,h)(t)$ and the auxiliary basis functions $w_a(x)$ and $w_b(x)$. Here, the basis functions are considered to be fixed and the collocation points defined as

$$x_i = \begin{cases} a, & i = -N-1 \\ \varphi(ih), & i = -N \dots N \\ b, & i = N+1 \end{cases}$$

With the collocation points defined as above, we set the approximate solution u of (8) as

$$u_N(x) = u_{-N-1}w_a(x) + \sum_{j=-N}^{N} u_j S(j,h)\big(\{\varphi\}^{-1}(x)\big) + u_{N+1}w_b(x) \tag{10}$$

and the integral in (8) becomes

$$\int_a^x k(x,t)u(t)dt$$
$$= K_V[w_a](x_k)u_{-N-1} + h\sum_{j=-N}^{N} k(x,t_j)\varphi'(jh)u_j J(j,h)(x_k)$$
$$+ K_V[w_b](x_k)u_{N+1}$$
$$+ Ohe^{-\frac{\pi d}{h}} \tag{11}$$

Noting that $S(j,h)(\{\phi\}^{-1}(x_k) = S(j,h)(\{\phi\}^{-1}(\varphi(kh)) = S(j,h)(kh) = \delta_{kj}$.
with

$$K_V[f](x) = h\sum_{j=-N}^{N} k(x,t_j)\varphi'(jh)f(x_j)J(j,h)(x). \tag{12}$$

and

$$J(j,h)(x) = \left(\frac{1}{2} + \frac{1}{\pi}Si\left(\pi\frac{\varphi^{-1}(x)}{h} - j\pi\right)\right) \tag{13}$$

Using (10) and (11) in (8), we obtain the collocation formula as

$$\{w_a(x_k) - K_V[w_a](x_k)\}u_{-N-1} + \sum_{j=-N}^{N} \delta_{kj} - hk(x_k,t_j)\sigma'(jh)J(j,h)(x_k)u_j$$
$$+ \{w_b(x_k) - K_V[w_b](x_k)\}u_{N+1} = g(x_k) \tag{14}$$

representing a $(2N+3)$ x $(2N+3)$ system of linear equations, which can be expressed more compactly as

$$(E_n - V_n)u_n = g_n \tag{15}$$

with

$$E_n = w_a(x_k) + \sum_{j=-N}^{N} S(j,h)(\{\phi\}^{-1}(x_k)) + w_b(x_k)$$
$$V_n = K_V[w_a](x_k) + h\sum_{j=-N}^{N} k(x_k,t_j)\phi'(jh)J(j,h)(x_k)(+ K_V[w_b](x_k)$$
$$g_n = [g(a), g(x_{-N}), \dots, g(x_N), g(b)]^T$$
$$u_n = [u_{-N-1}, u_{-N}, \dots u_N, u_{N+1}]^T$$

By solving the above system of equations for u_n and using the result in (10), we obtain the approximate solution $u_N(x)$ to (8).

3.0 The Variable Transformation Function

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We will employ in this procedure the variable transformation defined by

$$x = \varphi(t) = \frac{(b-a)}{2}\operatorname{erf}(t) + \frac{(b+a)}{2} \quad (16)$$

known as the translated error function
with its inverse

$$t = \varphi^{-1}(x) = erf^{-1}\left(\frac{2x-b-a}{b-a}\right). \quad (17)$$

describes a mapping $\varphi(-\infty) = a$ and $\varphi(\infty) = b$ with the sinc points given by $x_j = \varphi(jh), j = -N, \ldots, N$.
Equation (16) reduces to

$$x = \sigma(t) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}(t) \quad (18)$$

for $a = 0$ and $b = 1$.
Let D be a bounded and simply connected domain, then $H^{\infty}(D)$ denote the family of functions $f \in Hol(D)$ such that $\|.\|_{H^{\infty}(D)}$ is finite
where

$$\|f\|_{H^{\infty}(D)} = \sup_{z \in D}|f(z)| \quad (19)$$

Furthermore, given a constant $d > 0$, we define
$D_d = \{z \in C: |im\, z| < d\}$ to denote a strip region of width $2d$, when incorporated with (18), this definition should be considered on the translated domain

$$\varphi(D_d) = \left\{z \in C: \left|\arg\left(\frac{2x-b-a}{b-a}\right)\right| < d\right\}. \quad (20)$$

4.0 Numerical Solutions of Volterra Integral Equations of the Second Kind

For these examples, we will find the approximate result and compute the maximum error in order to determine the accuracy of the method.

Example 1
In this example we solve equation (8) with
$k(x,t) = \sin(x-t),\ g(x) = x, 0 \le x \le 1$, Ren *et al* (1999)
which has exact solution $u(x) = x + \frac{x^3}{6}$.
We will first apply Sinc collocation method using SE formula, with

$h = \sqrt{\pi d/\alpha N}$, choosing $\alpha = 1$and $d = \frac{\pi}{2}$.
using the SE variable transformation

$$x = \phi(t) = \frac{a + be^t}{1 + e^t} \tag{21}$$

with inverse

$$t = \phi^{-1}(x) = \log\left(\frac{x - a}{b - x}\right) \tag{22}$$

and derivative

$$\phi'(t) = \frac{(b - a)e^t}{(1 + e^t)^2}. \tag{23}$$

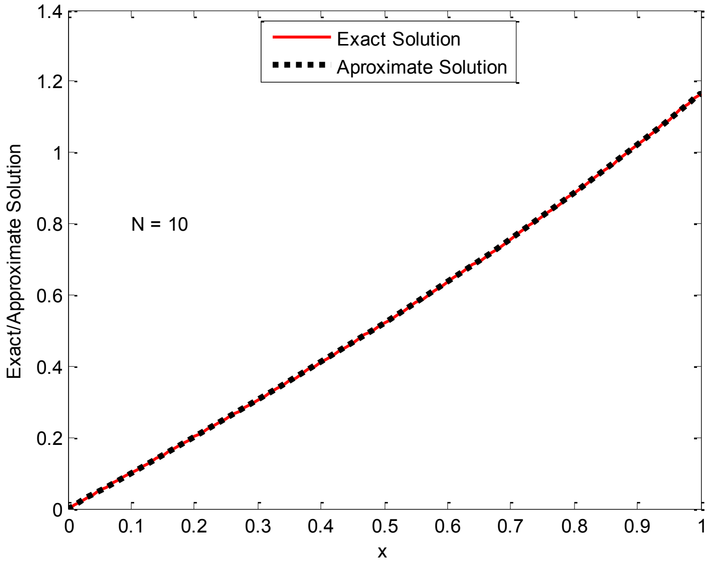

Fig. 1: Exact solution and approximate solution for Example 1 using SE

4.1 Presentation of Results

The approximate solution in Fig. 1 has a maximum deviation of $1.9857\text{x}10^{-6}$ from exact solution at x = 0.80.

We will implement Error Function Transform (EFT) formula, for the solutions of Example 1. For this example, we choose $\alpha = 1$, $d = \frac{\pi}{2}$ so that $h = \left(\frac{\pi}{N}\right)^{\frac{2}{3}}$.

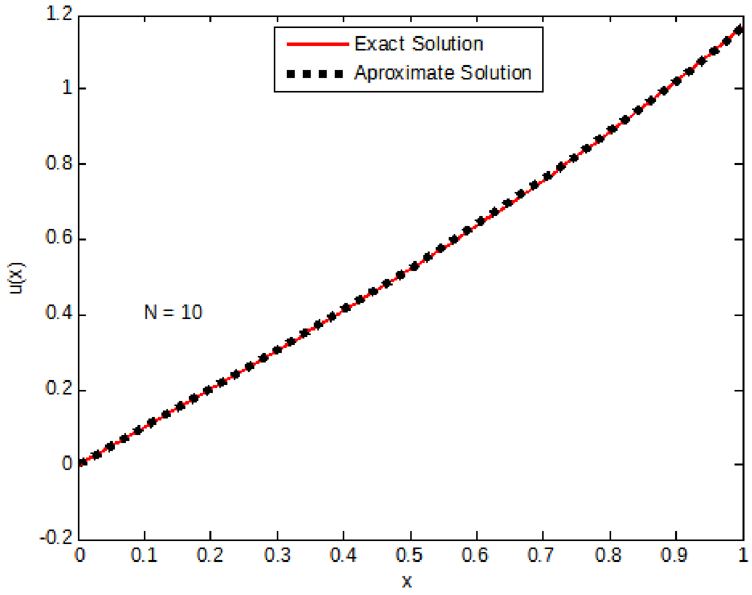

Fig. 2: Exact solution and approximate solution for Example 1 using EFT

The approximate solution in Fig. 2 has a maximum deviation of 2.4746×10^{-5} from exact solution.

Example 2

In this example, we solve equation (8) with $k(x,t) = t - x$, $g(x) = x - x^2 + \frac{x^2}{6} - \frac{x^4}{12}$, which has the exact solution $x - x^2$. Rashidinia and Zarebnia (2008).

We will use Sinc collocation with SE formula.

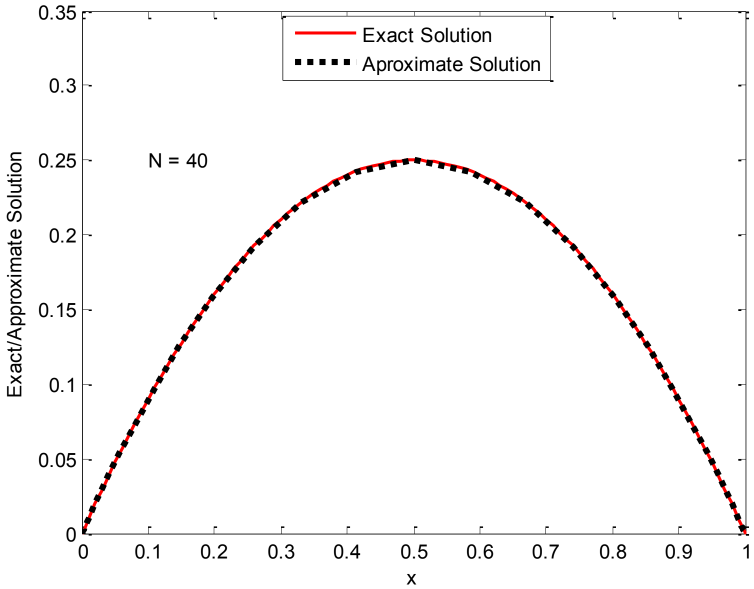

Fig. 3: Exact solution and approximate solution for Example 2 using SE formula

The maximum absolute error of Example 2 as shown in Fig 3 is 2.8367×10^{-11} at N = 40 and occurring at x = 0.5.

We will now implement EFT formula for example 2.

The parameters for this purpose are $\alpha = 1$, $d = \frac{\pi}{2}$ so that $h = \left(\frac{\pi}{N}\right)^{\frac{2}{3}}$.

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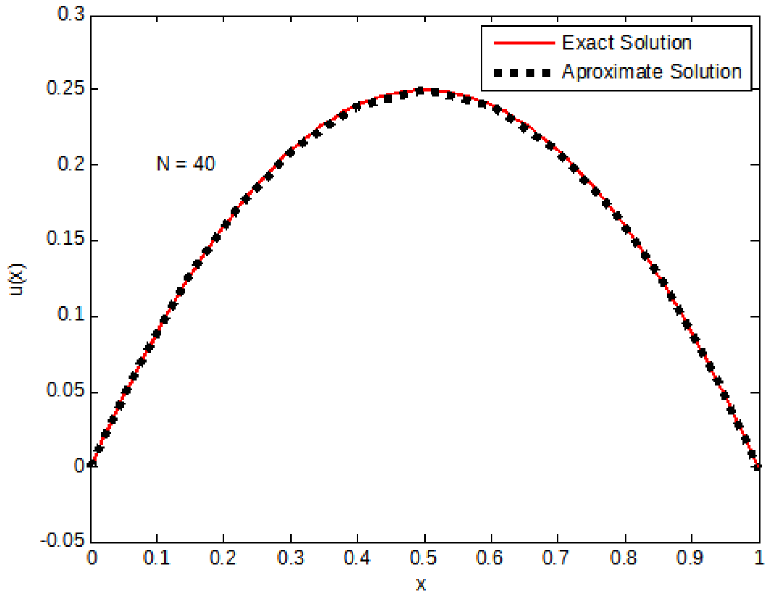

Fig. 4: Exact solution and approximate solution for Example 2 using EFT

The maximum absolute error for Example 2 using EFT formula at N = 40 is 2.1431×10^{-13} as seen in Fig. 4 above. The decrease of maximum absolute error as N increases, exhibiting the decay propriety associated with Sinc methods.

4.2 Analysis of results based on SE formula and EFT formula

After showing the implementations of the collocation scheme based on SE and EFT formulas, we will in this section be concerned with analysis of the performance of the method on Examples 1 and 2. Let $|E_N(h(\phi))|$ and $|E_N(h(\varphi))|$ represent the maximum absolute error at Sinc points for the approximated values of $u(x)$ based on SE and EFT formulas respectively. The table below illustrates the differences in decay rates of the maximum absolute error in the approximate solutions of Examples 1 and 2.

Table 1: Comparison of maximum absolute error for Example 1 and Example 2 between SE and Error Function Transform formulas.

			Example 1		Example 2	
N	$h(\phi)$	$h(\varphi)$	$\lvert E_N(h(\phi))\rvert$	$\lvert E_N(h(\varphi))\rvert$	$\lvert E_N(h(\phi))\rvert$	$\lvert E_N(h(\varphi))\rvert$
10	0.7025	0.3447	$1.9857\text{x}10^{-6}$	$2.4746\text{x}10^{-5}$	$5.5913\text{x}10^{-6}$	$1.1184\text{x}10^{-4}$
20	0.4967	0.2070	$2.6239\text{x}10^{-8}$	$5.6215\text{x}10^{-8}$	$4.0909\text{x}10^{-8}$	$1.8154\text{x}10^{--7}$
30	0.4056	0.1515	$1.9848\text{x}10^{-9}$	$2.4112\text{x}10^{-10}$	$8.1749\text{x}10^{-10}$	$2.3443\text{x}10^{-10}$
40	0.3512	0.1208	$2.1996\text{x}10^{-10}$	$1.4491\text{x}10^{-12}$	$2.8267\text{x}10^{-11}$	$2.1431\text{x}10^{-13}$
50	0.3112	0.1101	$3.1784\text{x}10^{-11}$	$1.1768\text{x}10^{-14}$	$1.4035\text{x}10^{-12}$	$2.2204\text{x}10^{-16}$

5.0 CONCLUSION

Table 1 shows the differences in the maximum absolute error for Sinc colocation methods for the solution of Volterra integral equations of the second kind based on SE formula and EFT formula for Examples 1 and 2 respectively. As observed in both examples, EFT formula improved in accuracy over SE formula as N increases based on the measurement of their maximum absolute errors. In summary, SE performs better than EFT for smaller N while EFT performs better than SE for large N.

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