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Computation of Inverse of N − Square Matrices Using Crammer's Rule for the Solution of Linear Equations

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ABSTRACT

A method for computing the inverse of square matrices is discussed in this work. The aim is to develop an alternative method for the computation of inverse A^{-1} of an $n \times n$ matrix. By employing Crammer's rule used for the solution of linear system of equations, an inverse of the matrix A is obtained as a coefficient of a catalytic column vector of a supposed solution of linear system of equation $\mathbf{Ax} = b$. However, a separate computation is not needed for the adjoint of the matrix since this process is absorbed by the Crammer's rule. Therefore, practical illustrations are given to demonstrate the applicability of the method.

Keywords: *Crammer's rule, Cofactors, rank, inverse, linear system of equations*
MSC 2010: 15A06, 15A09

INTRODUCTION

This study considered the development of an alternative method for the computation of inverse A^{-1} of an $n \times n$ matrix A. Matrix inversion is important in the study of solutions of linear system of equations, noting that for the linear system of equations

$$Ax = b \qquad (1)$$

the solution

$$x = A^{-1}b \qquad (2)$$

depends on the existence of the inverse of the coefficient matrix A in (1).

There are several methods developed for the inversion of matrices which are discussed in the following papers (Tian, Guo, Ren and Ai, 2014), (Smith and Powell, 2011), (Thirumurugan, 2014), (Du Croz and Higham, 1992) (John, 2016). Among these methods, the method of cofactors and adjoint of matrices is mostly taught, even though there are questions bordering on the cost of computation. In the proposed

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procedure, the method of cofactors is employed alongside the Crammer's rule. Since our interest is in computing the inverse of the matrix and not the solution of an equation itself, the entries of the column vectors in b for this procedure are unknown constants with catalytic impact in the overall computation. This study gives some basic definitions of terms required for our subsequent discussions on the subject. The main results are presented with applications of the procedure illustrated.

Basic Definitions

Definition 1

Let $A = (a_{ij})$ be an $n \times n$ matrix, the determinant of $(n-1) \times (n-1)$ matrix obtained by removing the rth row and the sth column from A is called the minor of the entry a_{ij} of and denoted by M_{rs}

Definition 2

The number A_{ij} defined by is called a cofactor of the entry a_{ij} of A.

Definition 3

Determinant of an $n \times n$ Matrix (Johnson, Riess and Arnold, 1998)

For any $n \times n$ matrix $A = (a_{ij})$, the determinant of A is the number det (A), defined by

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{n+1}a_{1n}\det(M_{1n})$$

$$= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j})$$

$$= \sum_{j=1}^{n} a_{1j} A_{1j}$$

where A_{1j} is the cofactor of a_{1j}, $1 \leq j \leq n$.

Definition 4

Crammers Rule (Kreyszig, 2011)
Let $A\mathbf{x} = b$ be the linear system of equations defined in equation (1) with unknown variables $x_1, x_2, \ldots, x_n$ and $a_{11}, a_{12}, \ldots, a_{nn}$ denoting the entries of the coefficient matrix A. Let $\det(A) \neq 0$, the determinant of A be denoted by D_0, then the solution of the above linear system of equations (1) is given by

$$x_1 = \frac{D_1}{D_0}, x_2 = \frac{D_2}{D_0}, \ldots, x_n = \frac{D_n}{D_0}$$

where $D_k(k = 1,2, \ldots n)$ is the determinant obtained from D by replacing in D the k*th* column by the column with the entries $b_1, b_2, \ldots b_n$.

MAIN RESULT

Proposition 1

Let $A\mathbf{x} = b$ be the linear system of equations (1) where A is a $n \times n$ matrix and $\boldsymbol{b}$ is $n \times 1$ column vector, then equation (1) has a solution if there exists A^{-1} such that $\mathbf{x} = A^{-1}b$.

Proof:

Assuming A^{-1} exists, then A consists of linearly independent columns with det(A)$\neq 0$; hence the $\text{rank}(A) = \text{rank}(A{:}\,b) = n$, so that the condition for existence of a unique solution x of (1) is satisfied.

Conversely, if $\text{rank}(A) < \text{rank}(A{:}\,b)$, then A has at least one linearly dependent row/column such that det (A) = 0 and A is singular. Therefore, $A^{-1}b$ is undefined.

Furthermore, if A^{-1}exists, from (1)

$$A^{-1}Ax = A^{-1}b$$

$$1x = A^{-1}b$$

since $A^{-1}A = I.$

And the solution of (1) can be expressed as

$$x = A^{-1}b$$

or equivalently

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad (3)$$

in terms of inverse of the coefficient matrix A and column vector b respectively.

Proposition 2

Let $x_k\,(k = 1,2, \ldots, n)$ be the solution of (1) obtained by Crammer's rule, then

(i) $$x_k = \sum_{j=1}^{n} D_{jk}\, b_j$$

(ii) $x = Db$

where

$$D = \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{n1} \\ D_{12} & D_{22} & \cdots & D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1n} & D_{2n} & \cdots & D_{nn} \end{pmatrix}.$$

Proof:

(i) Let $x_k, (k = 1,2, \ldots, n)$ denote the n-tuple vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and A_{jk} the cofactors corresponding to a_{jk} of the coefficient matrix A, then by Crammer's rule and definition 3,

$$x_1 = \frac{D_1}{D_0} = \frac{A_{11}b_1 + A_{21}b_2 + A_{31}b_3 + \cdots + A_{n1}b_n}{D_0} = \frac{1}{D_0}\sum_{j=1}^{n} A_{j1}b_j$$

$$x_2 = \frac{D_2}{D_0} = \frac{A_{12}b_1 + A_{22}b_2 + A_{32}b_3 + \cdots + A_{n2}b_n}{D_0} = \frac{1}{D_0}\sum_{j=1}^{n} A_{j2}b_j$$

$$\vdots$$

Hence for $k = 1,2, \ldots , n$

$$x_k = \frac{1}{D_0}\sum_{j=1}^{n} A_{jk}\, b_j = \sum_{j=1}^{n} D_{jk}\, b_j = \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{n1} \\ D_{12} & D_{22} & \cdots & D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1n} & D_{2n} & \cdots & D_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad (4)$$

(ii) By (i) above,

$$\mathbf{x} = \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{n1} \\ D_{12} & D_{22} & \cdots & D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1n} & D_{2n} & \cdots & D_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = D\boldsymbol{b} \qquad (5)$$

Proposition 3
The matrix D in proposition 2 is the inverse of A

Proof:
Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

be the solution of the linear system of equations (1), then

$$\mathbf{x} = A^{-1}b = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{n1} \\ D_{12} & D_{22} & \cdots & D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1n} & D_{2n} & \cdots & D_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = D\boldsymbol{b}$$

Therefore,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{n1} \\ D_{12} & D_{22} & \cdots & D_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{1n} & D_{2n} & \cdots & D_{nn} \end{pmatrix}$$

and $A^{-1} = D$ (6)

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APPLICATIONS

In this section, we compute D using the Crammer's rule for the solution of linear system of equations.

For any $n \times n$ matrix A with $D_0 \neq 0$,
Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

be column vectors such that $A\mathbf{x} = \mathbf{b}$ is consistent. Since our interest is to find the matrix D, choosing the entries of $\mathbf{b}$ as unknown constants do not affect the outcome of the computation since is only a factor of $A^{-1}b$.

Example 1

Find the inverse of the matrix using Crammer's rule

$$\begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$$

Solution

Let $\quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$A\mathbf{x} = \boldsymbol{b} = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

is consistent since

$$\det(A) = D_0 = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3 \neq 0.$$

Expanding the determinants D_1 and D_2 along the $\mathbf{b}$ column, noting the 'place' sign of the elements,

$$x_1 = \frac{D_1}{D_0} = \frac{\begin{vmatrix} b_1 & 5 \\ b_2 & 4 \end{vmatrix}}{3} = \frac{4b_1 + (-5)b_2}{3}$$

$$x_2 = \frac{D_2}{D_0} = \frac{\begin{vmatrix} 2 & b_1 \\ 1 & b_2 \end{vmatrix}}{3} = \frac{(-1)b_1 + 2b_2}{3}.$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4b_1 - 5b_2 \\ -b_1 + 2b_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Therefore, the inverse of A is the matrix

$$D = \frac{1}{3}\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}.$$

The adjoint of A is the matrix $\begin{pmatrix} 4 & -5 \\ -1 & 2 \end{pmatrix}$ with $A_{11} = 4$, $A_{12} = -1$, $A_{21} = -5$, $A_{22} = 2$ as cofactors of a_{11}, a_{12}, a_{21} and a_{22} respectively.

Example 2

Obtain the inverse of the matrix $A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 6 \\ 1 & 2 & 7 \end{pmatrix}$ using Crammer's rule.

Solution

$$|A| = D_0 = -12,$$

Let

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

such that for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$A\mathbf{x} = b$ is consistent.

Then from Crammer's rule,

$$x_1 = \frac{D_1}{D_0} = \frac{1}{D_0}\begin{vmatrix} b_1 & 4 & 3 \\ b_2 & 5 & 6 \\ b_3 & 2 & 7 \end{vmatrix} = -\frac{1}{12}(23b_1 - 22b_2 + 9b_3)$$

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$$x_2 = \frac{D_2}{D_0} = \frac{1}{D_0}\begin{vmatrix} 1 & b_1 & 3 \\ 2 & b_2 & 6 \\ 1 & b_3 & 7 \end{vmatrix} = -\frac{1}{12}(-8b_1 + 4b_2 - 0b_3)$$

$$x_3 = \frac{D_3}{D_0} = \frac{1}{D_0}\begin{vmatrix} 1 & 4 & b_1 \\ 2 & 5 & b_2 \\ 1 & 2 & b_3 \end{vmatrix} = -\frac{1}{12}(-b_1 + 2b_2 - 3b_3).$$

$$\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\frac{1}{12}\begin{pmatrix} 23b_1 - 22b_2 + 9b_3 \\ -8b_1 + 4b_2 - 0b_3 \\ -b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

$$= -\frac{1}{12}\begin{pmatrix} 23 & -22 & 9 \\ -8 & 4 & 0 \\ -1 & 2 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Therefore, the inverse of A is given by

$$D = -\frac{1}{12}\begin{pmatrix} 23 & -22 & 9 \\ -8 & 4 & 0 \\ -1 & 2 & -3 \end{pmatrix}.$$

Example 3

Find the adjoint of the matrix given by

$$A = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 5 & 8 \\ 2 & 0 & -3 \end{pmatrix}$$

Solution

$$D_0 = \det(A) \neq 0.$$

Applying Crammer's rule,

$$x_1 = \frac{D_1}{D_0} = \frac{1}{D_0}\begin{vmatrix} b_1 & -2 & 4 \\ b_2 & 5 & 8 \\ b_3 & 0 & -3 \end{vmatrix} = \frac{1}{D_0}(-15b_1 - 6b_2 - 36b_3)$$

$$x_2 = \frac{D_2}{D_0} = \frac{1}{D_0}\begin{vmatrix} 3 & b_1 & 4 \\ 1 & b_2 & 8 \\ 2 & b_3 & -3 \end{vmatrix} = \frac{1}{D_0}(19b_1 - 17b_2 - 20b_3)$$

$$x_3 = \frac{D_3}{D_0} = \frac{1}{D_0}\begin{vmatrix} 3 & -2 & b_1 \\ 1 & 5 & b_2 \\ 2 & 0 & b_3 \end{vmatrix} = \frac{1}{D_0}(-10b_1 - 4b_2 + 17b_3).$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{D_0}\begin{pmatrix} -15b_1 - 6b_2 - 36b_3 \\ 19b_1 - 17b_2 - 20b_3 \\ -10b_1 - 4b_2 + 17b_3 \end{pmatrix}$$

$$= \frac{1}{D_0}\begin{pmatrix} -15 & -6 & -36 \\ 19 & -17 & -20 \\ -10 & -4 & 17 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

$$\mathrm{x} = \frac{1}{\det(A)} \cdot adj(A) \cdot b$$

Therefore, the adjoint of A is given by

$$adj(A) = \begin{pmatrix} -15 & -6 & -36 \\ 19 & -17 & -20 \\ -10 & -4 & 17 \end{pmatrix}.$$

CONCLUSION

The purpose of this work was the computation of the inverse of the matrix and not the solution of an equation itself. The column vectors in b for the procedure were unknown constants with catalytic impact in the overall computation. The study offered some basic definitions of terms required for the discussion on the subject. However, the main results were presented and the applications of the procedure were illustrated. The alternative method for the computation of inverses of $n \times n$ matrices was presented with the cofactors computed as parts of the Crammer's rule. In this method, a separate computation is not needed for the adjoint of the matrix since this process is absorbed by the Crammer's rule. This procedure has been illustrated to demonstrate its efficiency in this study.

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