VARIATIONAL SOLUTION OF CRITICAL NORMAL STRESS DISTRIBUTION OF FOOTING ON SLOPE

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ABSTRACT

A mathematical technique was advanced for investigating the normal stress distribution failure of soil foundations. The stability equations were obtained using the limit equilibrium (LE) conditions. The additions of vertical, horizontal and rotational equilibria were transformed mathematically with respect to the soil shearing strength, leading to the derivation of the equation of the functional Q, and two integral constraints. In the mathematical method employed, the stability analysis was transcribed as a minimization problem using the calculus of variations. Generally, no constitutive law beyond the Coulomb's yield criterion was incorporated in the formulation; consequently, no constraints are placed on the character of the criticals except the overall equilibrium of the failing soil section. The critical normal stress distribution, δmin, and consequently the load, Qmin, determined as a result of the minimization of the functional are the smallest stress and load parameters that can cause failure. In other words, for a soil with strength parameters c, ø, ɤ, and footing with geometry B, H, when stress $\delta < \delta$min (c, ø, ɤ, B, H) and load Q < Qmin (c, ø, ɤ, B, H) foundation is stable. Otherwise, the stability would depend on the constitutive character of the foundation soil. In the mathematical method employed, the stability analysis is transcribed as a minimization problem using the calculus of variations.

Key Word: Cohesion, Internal Friction, Vertical load, Stress distribution, Rupture surface, Shear stress, Failure, Coordinate transformation, Polar coordinates.

INTRODUCTION

Many of the problems encountered in soil Mechanics and Foundation Engineering Designs are the extreme-value type. These problems include the stability of slopy soil, the bearing capacity of foundations both on horizontal, adjacent to slopy soil and on slopy soil, the limiting forces (active-Pa and passive Pp) acting on retaining structures like retaining walls, dams, sheet pile walls and others. All problems of the types mentioned above can be solved within the framework of the limiting equilibrium (LE) approach. This approach which considers the overall stability of a "test body" bounded by soil surface [$y(x)$] and slip surface [$y(x)$] according to Akubuiro (1991) is based on the following three concepts.

(a) Satisfaction of failure criteria $S = f(\delta)$ along the slip surface, y(x) over which $\tau(x)$ and δ(x) constitute the shear and normal stresses distribution.

(b) Satisfaction of all equilibrium equations for the test body (vertical, horizontal and rotational equilibria).

(c) Extremization of the factor S with respect to two unknown functions y(x) and δ (x). Thus S is considered to be function of these (y (x) and δ (x) functions.

The extreme value Sex is defined as:

$$S_{ex} = Extr\ S\ [y\ (x)\ ,\ \delta\ (x)\ ] ---------------------- 1.1$$

However, the determination of the bearing capacity of soil and associated critical rupture surface and normal stress condition along the surface remains one of the most important problems of engineering soil mechanics. Several approaches to this problem have evolved over the years.

One of the early sets of bearing capacity equations was proposed by Terzaghi in 1943. These equations by Terzaghi used shape factors noted when the limitations of the equation were discussed. These equations were produced from a slightly modified bearing capacity theory developed by Prandtl in 1920 from using the theory of plasticity to analyze the punching of rigid base into a softer (soil) material (Bowles, 1997; Chukwueze, 1990). Another method which has been widely used though equally misleading involves the determination of the bearing capacity by the plate loading test at a given work site.

Accordingly, Prandtl identified zones in the metal at failure as follows:

(a) A wedge zone under the loaded area pressing the material downward as a unit.

(b) Two zones of all-radial failure planes bounded by a logarithmic spiral curve.

(c) Two triangular zones forced by pressure upward and outward as two independent units.

Although efforts were made by Hansen, Meyerhorf, Vesic and many other researchers (Garg, 2005) to present more encompassing and dependable solution, it was Terzaghi (Bowles, 1997) who developed the first rational and practical approach to this problem. The method involves three determinant factors:

(a) the soil unit weight, *r*.

(b) the effect of surcharge, *q* or applied load *Q*.

(c) the strength parameters of the soil, therefore, it is more comprehensive than any other approach before it.

Meyerhoff (1951), Bowles (1997), Garg (2005), Smith G. and Smith Ian (1998), Feda (1961), Terzaghi and Peck (1948), Terzaghi (1943) had also obtained by a technique similar to that of Terzaghi's approximate solutions by including shape and depth factors for plastic equilibrium of footing by assuming failure mechanism, and like Terzaghi expressed results with bearing capacity factors.

Hitherto, none of the above two parameters has been mathematically quantified and so bearing capacity calculations have been based on various assumed rupture lines and normal stress distributions. The existing methods therefore, differ from one another in the assumptions about the character of the functions $y(x)$ and Most of the assumptions are motivated by the available plasticity solutions for idealized cases. The resulting solutions, therefore, contain errors of unknown magnitude.

This work therefore attempts to further advance the solution to the stability problem by formulating the stability equations using the limiting equilibrium conditions, transcribing the problem as a minimization problem in the calculus of variations and then determining the normal stress distribution along the failure surface with the basic assumption that the foundation is on a slope. With the normal stress distribution at failure and the rupture surface mathematically defined, so far, the determination of the bearing capacity of soil and associated critical rupture surface and normal stress condition along the surface remains one of the most important problems of engineering soil mechanics. However the objective of this research work is to basically determine the critical stress distribution along this plane of failure by employing a more mathematical approach to finding solutions to this problem.

Basic Principles of Variational Calculus

The calculus of variations deals with the problem of maxima and minima (Swokowski, 1991 and Elsgolts, 1977). But while in the ordinary theory of maxima and minima, the problem is to determine those values of the independent variables for which a given function of these variables takes a maximum or minimum value, in calculus of variations, definite integrals involving one or more unknown functions are considered and it is required to determine those unknown functions that the definite integrals shall take a maximum or minimum value (Pars 1962; Pipes and Harvill 1971). The definite integrals here are called functionals.

The Euler-Lagrangian Equation

The basic thrust of the Euler-Lagrangian equation is stated in analytical terms as Elsgolts (1977). Given that there exists a twice differentiable function Y = y(x) satisfying the conditions $y(x_1) = y_1$, $Y(x_2) = Y_2$, and which renders the functional

$$J = \int_{x_1}^{x_2} f\ (x,\ y,\ y^1)\ dx --------2.1$$

a minimum, what is the differential equation satisfied by y(x)? The constants x_1, x_2, y^1, y^2 are supposedly given and f is a function of the arguments x, y, y^1 which is twice differential with respect to any, or any combination of them (Robert,1952).

We denote the function that extremizes equation (2.1) by y(x) and proceed to form the one parameter family of "comparison" functions $y(\alpha, x)$ defined by;

$$Y\ (\alpha,\ x) = y\ (o,\ x) + \alpha\eta(x) ------------- 2.2$$

Where $\eta(x)$ is an arbitrary differentiable function for which $\eta(x_1) = \eta(\alpha_2) = 0$ --------2.3

and α is the parameter of the family. Now replacing y and Y^1 in Equation 2.1 by y(x) and $y^1(x)$ respectively, we form the integral

$$J = \int_{x_1}^{x_2} f \ (x, y, y^1) \, dx - - - - - - - - 2.4$$

where for a given function η(x), the above integral is clearly a function of the parameter α.

The argument Y^1 is given, through equation 2.2 by $Y^1 = Y^1 \ (\alpha, x) = Y^1 \ (o, x) + \alpha \eta^1(x) - - - - 2.5$ certainly, the integral Equationn 2.4 is minimum at α = o and is equivalent to replacing Y and Y^1 respectively with Y(x) and $Y^1(x)$. Also from elementary calculus (Stroud, 1995) the necessary condition for a minimum is that the vanishing of the first derivative of J with respect to α must hold for α = o, thus;

$$\frac{\partial J}{\partial \alpha} = J^1(\alpha) \ - - - - - - - - - - - - - - - - - 2.6$$

$$\frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f \ \left(y, y^1, x\right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial Y^1} \cdot \frac{\partial Y^1}{\partial \alpha} \right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial Y} \cdot \eta + \frac{\partial f}{\partial Y^1} \cdot \eta^1 \right] dx - - - - 2.7$$

Since setting α = 0 is equivalent to replacing (Y, Y^1) by (Y(x), $Y^1(x)$), we have according to equation 2.7,

$$J^1(o) = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y} \eta + \frac{\partial f}{\partial Y^1} \eta^1 \right) dx = 0 - - - - - - 2.8$$

Integrating by parts, the second term in the integral we obtain

$$J^1(o) = {\partial f}/{\partial Y^1} \ \eta \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[{\partial f}/{\partial Y} - \frac{d}{dx} \left({\partial f}/{\partial Y^1} \right) \right] \eta dx \ - - - - - - - - 2.9$$

As a result of equation 2.3

$$\frac{\partial f}{\partial Y}\,\eta\Big|_{x_1}^{x_2} = 0 ----------2.10$$

$$and\ \ J^1(0) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y^1}\right)\right]\eta dx\Big|_{\alpha = 0} = 0-----2.11$$

The only way the above equation 2.9 can equal zero since (x) is zero only at end points is for the function.

$$\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\partial f \big/ \partial Y^1\right) = 0 ---------2.12$$

This is the Euler-Lagrangian differential equation which is the necessary condition for the function J to have an extremum.

Basis for Parametric Representation

We proceed to show, however, that the extremizing relationship between a pair of variables x and y is the same, whether the solution is derived under the assumption that Y is a single-valued function of x or that a more general parametric representation is required to express the relationship between x and y. This we do by showing that the solution of the Euler Lagrange equation derived on the basis of the assumption of the single-valuedness of Y as a function of x satisfies also the system of Euler-Lagrangian equation derived on the basis of the parametric relationship between x and y (Meyerhoff, 1951).

Under the assumption that Y is a single – valued function of x, the functional to be minimized is given as recalled

$$J = \int_{x_1}^{x_2} f\ (x,\ y,\ Y^1)\ dx---------2.13$$

where y is required to have the values Y_1 and Y_2 at x = x_1 and x = x_2 respectively. If instead we use the parametric representation x = x(t) and y = y(t), where x(ti) = xi, y(ti) = yi for i =1, 2, the integral (2.13) is transformed through the relationships.

$$Y^1 = \frac{dy}{dx} = \frac{\bar{y}}{\bar{x}} ---------------------2.14$$

$$dx = \bar{x}dt ------------------------2.15$$

$$J = \int_{t_1}^{t_2} f\left(x, y, \frac{\bar{y}}{\bar{x}}\right) \bar{x}\,dt -------------2.16$$

But the Euler-Lagrangian equation corresponding to equation 2.13 is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right) = 0 --------------2.17$$

The system of Euler-langrangian equations associated with e.g. 2.16 is, if we write.

$$g\,(x, y, x, y) = f\,(x, y, y^1)\ \bar{x} -----------2.18$$

$$Y^1 = \frac{\bar{y}}{\bar{x}} -------------------------2.19$$

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial x}\right) = 0 -----------------2.20$$

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\left(\frac{\partial g}{\partial \bar{y}}\right) = 0 -----------------2.21$$

From equations 2.18, 2.19, 2.20, we obtain

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x}\ \bar{x} ------------------2.22$$

$$\frac{\partial g}{\partial x} = f - \bar{x}\,\frac{\partial f}{\partial y^1}\,\frac{\bar{y}}{\bar{x}^2} = f - y^1\,\frac{\partial f}{\partial y^1} ---------2.23$$

From equation 2.15, we have, after substituting in 2.22, and 2.23

$$\frac{d}{dt}\left(\frac{\partial g}{\partial x}\right) = \bar{x}\,\frac{d}{dx}\left(f - Y^1\,\frac{\partial f}{\partial Y^1}\right) = x\left[Y^1\left[\partial f/\partial Y^1 - \frac{d}{dx}\left(\frac{\partial f}{\partial Y^1}\right)\right] + \frac{\partial f}{\partial x}\right] -----2.24$$

Furthermore, differentiating equations 2.18 and 2.19 gives:

$$\frac{\partial g}{\partial Y} = \frac{\partial f}{\partial y}\ \bar{x} -----------------2.25$$

$$\frac{\partial g}{\partial Y} = x\,\frac{\partial f}{\partial Y^1}\,\frac{1}{\bar{x}} = \frac{\partial f}{\partial Y^1} ----------2.26$$

Thus, according to equations 2.14 and 2.15

$$\frac{d}{dt}\left(\frac{\partial g}{\partial y}\right) = \bar{x}\,\frac{d}{dx}\left(\frac{\partial f}{\partial Y^1}\right) \quad --2.27$$

combining the above result with 2.25; and 2.24 with 2.22 gives the following pair of equations:

$$\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial x}\right) = -\,\bar{y}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y^1}\right)\right] ----2.28$$

$$\frac{g}{y} - \frac{d}{dt}\left(\frac{\partial g}{\partial \bar{y}}\right) = \bar{x}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y^1}\right)\right] --------2.29$$

From this result, we conclude that any relationship, single-valued or not, that satisfies the Euler-Lagrangian Equation 2.13 derived on the basis of an assumed single-valued solution y = y(x) – satisfies also the system of equation 2.20 which derivation requires no assumption of single valuedness of y as a function of x.

MATHEMATICAL DERIVATIONS AND SOLUTION

A shallow strip foundation of width B is buried in a sloppy soil mass at a depth H as shown in fig. 3.1.

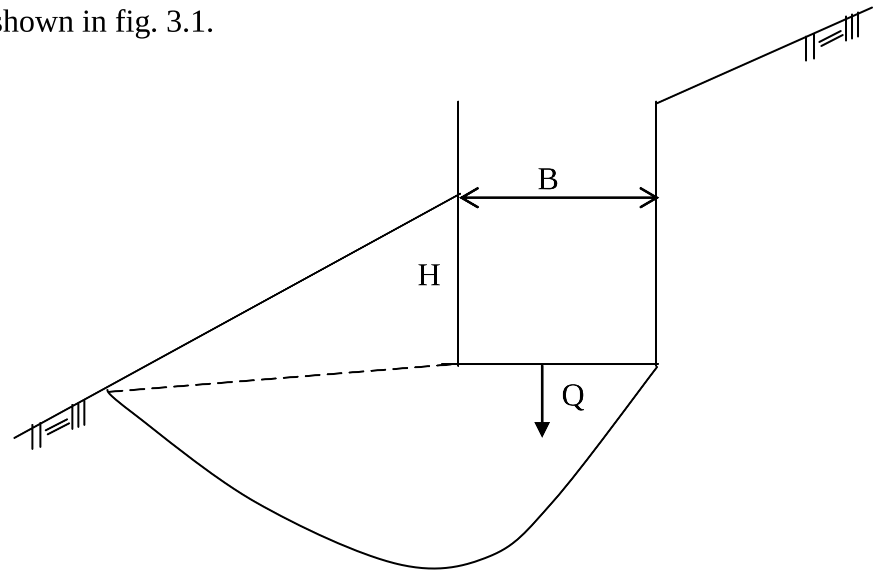

Fig. 3.1: Foundation buried in slopy soil mass

The soil mass is of semi-infinite extent and is homogeneous and isotropic. It has an effective unit weight r, and shear strength parameters C and ø (the cohesion and angle of internal friction respectively).

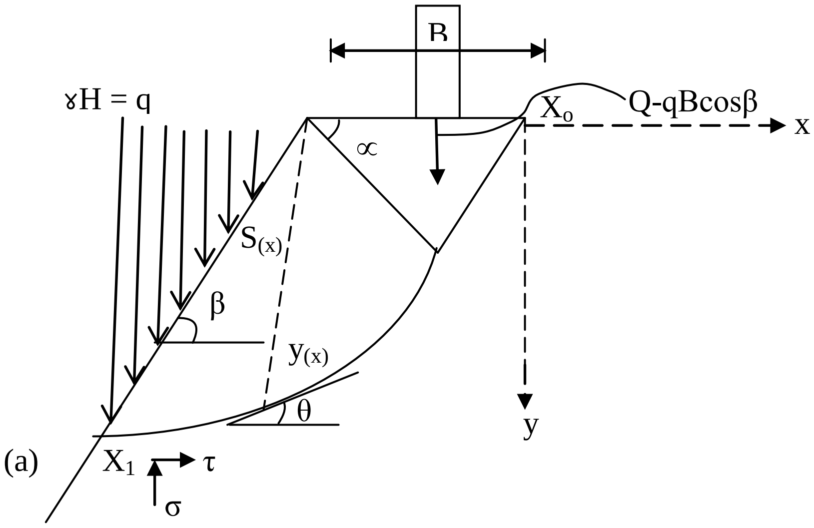

Fig. 3.2: Calculation scheme (S = arc length along y(x) and $\theta = \tan^{-1}(d_y/d_x)$

A mass of soil such as the one in fig. 3.2 is considered to be in a state of limiting equilibrium if:

(1) Coulomb's yield condition is satisfied along a potential rupture line Y(x) that smoothly connects one edge of the footing to the ground surface, thus

$$\tau(x) = C + \sigma(x) \tan \phi \text{ ---------------------------3.1}$$

where τ(*x*) and σ(*x*) are the shear and normal stress distributions along Y(x) respectively.

(2) The three equations of equilibrium-vertical, horizontal and rotational equilibrium-are satisfied for the sliding mass, thus:

(a) For vertical equilibrium, we have:

$$\sum_{i=1}^{n} Fiv = 0 \text{ ---------------------------3.2}$$

For vertical component of an equivalent force F_i which replaces the system of n forces in fig. 3.2. Resolving therefore all forces in the vertical direction and summing for vertical equilibrium, we have

$$Q - qB\cos\beta - \sum_{s}(\tau\sin\theta + \sigma\cos\theta)\, ds + \sum_{i=1}^{n} \gamma y dx + \int \gamma H_i\, d_x = 0 \quad \text{----3.3}$$

In the limit ds → 0 as x → 0, we have

$$Q - qB\cos\beta - \int_{s} (\tau\sin\theta + \sigma\cos\theta) d_s + \int_{X_o}^{X_1} \gamma_y\, d_x + \int_{X_o}^{X_1} \gamma H d_x = 0 \text{ -------------3.4}$$

On simplification, we have

$$Q - qB\cos\beta - \int_{s} (\tau\sin\theta + \sigma\cos\theta) d_s + \int_{X_o}^{X_1} \gamma (y + H) d_x = 0 \text{ -------------3.5}$$

(b) Similarly, for horizontal equilibrium,

$$\sum_{i-1}^{n} F_i h = 0 \text{ ----------------------------3.6}$$

$F_i h$ = horizontal component of force F_i. Resolving all forces horizontally, we have

$$\sum_{i=1}^{n} (\sigma\sin\theta - \tau\cos\theta)\, d_s = 0 \text{ -------------------3.7}$$

In the limit as ds → 0, we have

$$\int_s (\sigma\sin\theta - \tau\cos\theta)\, d_s = 0 \text{---------------------3.8}$$

(c) For rotational equilibrium, we have

$$\sum_{i=1}^{n} M_i = 0 \text{ --------------------------3.9}$$

$$\Rightarrow \sum_{i=1}^{n} Fiv.x + \sum_{i=1}^{n} Fih.y + \sum_{i=1}^{n} \gamma y.x\, dx$$

$$+ \sum_{i=1}^{n} \gamma H.x\; dx = 0 \text{ -------------3.10}$$

In the limit d_x → 0, d_s → 0, then

$$\int_s \left[(\tau \cos\theta - \sigma \sin\theta) y - (\tau \sin\theta + \sigma \cos\theta) x \right] d_s$$

$$+ \int_{X_0}^{X_1} \gamma (y + H) x \; d_x = 0 \text{ --------------3.11}$$

in which X_0 and X_1 are the end points y(x), s = the arc length along y(x) and α arc tan (d_y/d_x).

From equation 3.5,

$$Q - qB\cos\beta - \int_s (\tau \sin\theta + \sigma \cos\theta) d_s + \int_{X_0}^{X_1} \gamma (Y + H) \, d_x = 0,$$

we have on rearrangement

$$Q = qB\cos\beta + \int_s (\tau \sin\theta + \sigma \cos\theta) \, d_s - \int_{X_0}^{X_1} \gamma (y + H) d_x \text{ --------3.12}$$

In the limit as dQ → Q_{min}, it is intended to determine the equation of the function σ(x), the critical normal stress distribution without any prior assumption. In fact, the functions realizing the minimizing *Q* are those of y(x) and σ(x). If y(x) the rupture surface is taken as a logarithmic spiral curve, the present problem could be restated thus: Find the equation of the critical normal stress distribution σ(x) along y(x) and which realizes the minimum value of the functional Q defined by the integral equation 3.12 and subject to two integral constraint equations 3.8 and 3.11.

If the appropriate expression for σ(x) is determined, that coupled with that for y(x), it is therefore possible to easily use equation 3.12 to determine minimum Q (i.e. , the critical Q) identified with the bearing capacity.

Since both the Coulomb's yield criterion and the equilibrium conditions are simultaneously satisfied, we proceed thus:

Substitute equation 3.1 into equations 3.5, 3.8 and 3.11, we have

(a) For equation 3.5,

$$Q - qB\cos\beta - \int_s \left[(c + \sigma(x) \tan\phi) \sin\theta + \sigma(x) \cos\theta \right] d_s$$

$$+ \int_{X_0}^{X_1} \gamma \left\lfloor y_{(x)} + H \right\rfloor d_x \text{ --------------------3.13}$$

Making Q the subject of formula, and letting ψ = tan ϕ, the frictional coefficient of the soil, we have,

$$Q = qB\cos\beta + \int_s [(c+\sigma\psi)\sin\theta + \sigma\cos\theta]\, d_s$$

$$- \int_{X_0}^{X_1} \gamma(y+H)\, d_x \text{ ------------------------------ } 3.14$$

(b) For equation 3.8,

$$\int_s (\sigma\sin\theta - \tau\cos\theta) d_x = 0$$

$$But\ \tau = c + \sigma\tan\phi$$

$$\Rightarrow \tau = c + \sigma\psi \ \left(for\ \tan\phi = \psi\right)$$

$$\therefore \int_s [\sigma\sin\theta - (c+\sigma\psi)\cos\theta] d_s = 0 \text{ --------------------- } 3.15$$

$$\int_s [\sigma(\sin\theta - \psi\cos\theta) - c\cos\theta]\, d_s = 0 \text{ -------------------- } 3.16$$

(c) For equation 3.11,

$$\int_s [(\tau\cos\theta - \sigma\sin\theta)\ y - (\tau\sin\theta + \sigma\cos\theta)]\, d_s$$

$$+ \int_{X_0}^{X_1} \gamma x\,(y+H)\, d_x = 0 \text{ ------------- } 3.17$$

$$\int_s \{[(c+\sigma\tan\phi)\cos\theta - \sigma\sin\theta]\ y - [(c+\sigma\tan\phi)\sin\theta + \sigma\cos\theta]\ x\ \}$$

$$d_s + \int_{X_0}^{X_1} \gamma x\,(y+H)\, d_x = 0 \text{ ----------- } 3.18$$

But ψ = tan ϕ,

$$\int_s \{[(c+\sigma\psi)\cos\theta - \sigma\sin\theta]\ y - [(c+\sigma\psi)\sin\theta + \sigma\cos\theta]\, x\}^{d_s}$$

$$+ \int_{X_0}^{X_1} \gamma x\,(y+H)\, d_x = 0 \text{ -------------------------------- } 3.19$$

On rearranging, we have

$$\int_s \{[\sigma(\psi\cos\theta - \sin\theta) + c\,Cos\,\theta]\ y - [\sigma(\cos\theta + \psi\sin\theta) + c\sin\theta]\ x\}d_s$$

$$+ \int_{X_0}^{X_1} \gamma x (y+H)\, d_x = 0 \text{ ------------------- } 3.20$$

The foregoing formulation contains five parameters of the problem – C, ϕ, ɤ, B, H. In the following sections, a parametric transformation shall be carried out using non-dimensional quantities to reduce the number of problem parameters, a design to give series of advantages in both the construction of the solution and the presentation of the results.

Fundamental Assumptions

(i) The soil mass under study is homogenous and isotropic. That is to say that the properties of any soil element are assumed the same as the properties of the whole soil mass, irrespective of location or orientation of the soil element.

(ii) Coulomb's law is strictly valid;

$\tau = c + \sigma \tan\phi$ ------------------------3.21

(iii) On the imminence of failure, the failure mechanism satisfies the basic conditions of equilibrium thus vertical, horizontal and rotational simultaneously.

(iv) Failure of foundation is assumed to take place by the general shear mode and is characterized by the existence of well defined failure pattern which consists of footing to the ground surface. Failure is then accompanied by substantial rotation of the foundation and the final soil collapse occurs only on one side of the foundation.

(v) The ground surface is assumed to be slopy and the overburden pressure at foundation level is equivalent to a surcharge load $q^1_o = \gamma H \cos \beta$.

(vi) The load exerted on foundation is assumed to be vertical and symmetrical.

Boundary Conditions

Variational problems deal with two types of boundary conditions:

(a) Fixed and points such as $\bar{x}_o$ (fig. 3.2)

(b) End points that can slide along a prescribed curve. Their position is determined in such a way as to assume an external value of the functional. Since such points are not known in advance, a variational boundary condition known as the transverslaity condition has to be satisfied.

For the general shear mode of failure, the function y(x) has to satisfy the following end conditions in order to comply with it:

$\bar{y}_2 = \bar{y}\ (\bar{x} = \bar{x}_1) = 0$ --------------------3.22

$\bar{y}_0 = \bar{y}\ (\bar{x} = \bar{x}_0) = 0$ -------------------3.23

Using these conditions, we simplify the following expressions thus:

$$\int_{x_0}^{\bar{x}_1} \bar{y}^{1}\, d\bar{x} = \int_{x_0}^{\bar{x}_1} \frac{d\bar{y}}{d\bar{x}}\, d\bar{x} \text{ --------------------------3.24}$$

$$= \int_{x_0}^{\bar{x}_1} d\bar{y} \text{ --------------------------------3.25}$$

$$= [y] \Big|_{\bar{x}_0}^{\bar{x}_1} \text{ --------------------------------3.26}$$

$$= \left\{\bar{y}\left(\bar{x} = \bar{x}_1\right) - \bar{y}\left(\bar{x} = \bar{x}_0\right)\right\} = 0 \text{ ---------------3.27}$$

$$\therefore \int_{x_0}^{\bar{x}_1} \bar{y}^{1}\, d\bar{x} = 0 \text{ -----------------------------------3.28}$$

Similarly

$$\int_{x_0}^{\bar{x}_1} \bar{y}\, \bar{y}^{1}\, d\bar{x} \quad \int_{x_0}^{\bar{x}_1} \bar{y} \frac{d\bar{y}}{d\bar{x}}\, d\bar{x} \text{ ------------------------3.29}$$

$$= \int_{x_0}^{\bar{x}_1} \bar{y}\, d\bar{y} \text{ -------------------------------3.30}$$

$$= \left[\tfrac{1}{2}\, \bar{y}^{2}\right] \Big|_{\bar{x}_0}^{\bar{x}_1} \text{ --------------------------------3.31}$$

$$= \frac{1}{2}\left\{\bar{y}^{2}\left(\bar{x} = \bar{x}_1\right) - \bar{y}^{2}\left(\bar{x} = \bar{x}_0\right)\right\} = 0 \text{ ---------------3.32}$$

$$\therefore \int_{x_0}^{\bar{x}_1} \bar{y}\, \bar{y}^{1}\, d\bar{x} = 0 \text{ ------------------------3.33}$$

Finally, with regard to the parent problem, we notice that the location of the end points X_1 is not known in advance. This therefore demands of us to now apply the condition of transversality. In this particular case, the appropriate form of this condition is given as: (Swokowski, 1991)

$$\left(S - \bar{y}^{1} \frac{\partial s}{\partial y^{1}} - \sigma^{1} \frac{\partial s}{\partial \sigma^{1}}\right)\Big|_{\bar{x} = \bar{x}_1} = 0 \text{ ---------------3.34}$$

$$where \;\; \hat{\sigma}_1 = \bar{\sigma}\left(\bar{x} = \bar{x}_1\right) \text{ --------------------------3.35}$$

Non-dimensional Parametric Representation

To appropriately reduce the problem parameters to analytically manageable number and hence advantageously construct the solution, it is most convenient to introduce a set of non-dimensional parameters.

Define therefore the non-dimensional parameters as follows:, (32)

$$\bar{x} = \frac{x}{B}, \quad \bar{y} = \frac{y}{B}, \quad \bar{H} = \frac{H}{B} \quad \text{--------------------} 3.36$$

$$\bar{c} = \frac{c}{B_{\gamma}^{\;1}}, \quad \bar{\sigma} = \frac{\sigma}{B_{\gamma}}, \quad \bar{Q} = \frac{Q}{B^{2}_{\;\gamma}} \quad \text{--------------------} 3.37$$

$$\hat{c} = \frac{c_{\gamma}}{B} + \psi\bar{H} = \bar{c} + \psi\bar{H} \quad \text{--------------------} 3.38$$

$$\hat{\sigma} = \frac{\sigma_{\gamma}}{B} - \frac{2H}{B} = \bar{\sigma} - 2\bar{H} \quad \text{---------------------} 3.39$$

$$\hat{Q} = \frac{Q_{\gamma}}{B^{2}} - \frac{2H}{B} = \bar{Q} - 2\bar{H} \quad \text{----------------------} 3.40$$

The problem is now presented in terms of c, σ and Q. Now from the geometry of the rupture surface (fig. 3.2), it is easy to see that

$$ds = \frac{dx}{\cos\theta} \quad \text{----------------------------} 3.41$$

$$\bar{y}^{1} = \frac{dy}{dx} = \tan\theta \quad \text{-----------------------} 3.42$$

From the definition of the non-dimensional parameters

$$x = \bar{x}B, \; y = \bar{y}B, \; H = \bar{H}B \quad \text{----------------------} 3.44$$

$$c = \bar{c}\,\gamma\, B, \; \boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}\,\gamma\, B, Q = \bar{Q}\; B^{2}\;\gamma \quad \text{--------------------} 3.46$$

The parameters are then used to transform the problem equations 3.14, 3.16, and 3.20 as follows, (a) for equation 3.14, we have

$$Q = \gamma Hb\cos\beta + \int_{s} \left[\sigma\left(Cos\;\theta + \psi\sin\theta\right) + C\;\; Sin\;\theta\right]ds$$

$$- \int_{x_0}^{x_1} \gamma\left(y + H\right)dx$$

Introducing the appropriate parameter equations 3.44 and 3.46, we have

$$\gamma B^2 \overline{Q} = \gamma B\left(B\overline{H}\right)\cos\beta + \int_{x_0}^{\bar{x}_1} \left[\ \bar{\sigma}\ \left(\cos\ \theta + \psi \sin\theta\right) + \bar{c}\ \gamma\ B\sin\theta\ \right]$$

$$\frac{dx}{\cos\theta} - \int_{x_0}^{\bar{x}_1} \gamma\ \left(\bar{y}B + \overline{H}B\right)dx \text{ --------------------------------------} 3.47$$

$$\gamma B^2 \overline{Q} = \gamma B^2 H \cos\beta + \gamma\ B \int_{x_0}^{\bar{x}_1} \left[\bar{\sigma}\ \left(\cos\ \theta + \psi \sin\ \theta\right) + \bar{c} \sin\ \theta\right] \frac{dx}{\cos\theta}$$

$$-\gamma B \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right)dx \text{ --} 3.48$$

$$But\ dx = d\left(\bar{x}B\right) = Bd\bar{x} \text{ --} 3.49$$

Substituting 3.49 into 3.48

$$\gamma B^2 \overline{Q} = \gamma\ B^2 H \cos\ \beta + \gamma\ B^2 \int_{x_0}^{\bar{x}_1} \left[\bar{\sigma}\ \left(\cos\theta + \psi\ \sin\theta\right) + \bar{c} \sin\ \theta\right] \frac{d\bar{x}}{\cos\theta}$$

$$-\gamma B^2 \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right)d\bar{x} \text{ --} 3.50$$

Diving although by rB^2, we have

$$\overline{Q} = H \cos\ \beta + \int_{x_0}^{\bar{x}_1} \left[\bar{\sigma}\ \left(\cos\theta + \psi \sin\theta\right) + \bar{c} \sin\ \theta\right] \frac{d\bar{x}}{\cos\theta}$$

$$- \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right)d\bar{x} \text{ --} 3.51$$

$$Now\ \tan\ \theta = \frac{dy}{dx} = y^1 \text{ ----------------------------------} 3.52$$

$$y^1 = \frac{dy}{dx} = \frac{d\left(\bar{y}B\right)}{d\left(\bar{x}B\right)} \text{ ------------------------------------} 3.53$$

$$= \frac{Bd\bar{y}}{Bd\bar{x}} = \frac{d\bar{y}}{d\bar{x}} = \bar{y}1 \text{ ------------------------------------} 3.54$$

$$\therefore\ y^1 = \bar{y}^1 = \tan\alpha \text{ ------------------------------------} 3.55$$

From Eqn 3.51, we get

$$\overline{Q} = \overline{H}\cos\beta + \int_{x_0}^{\bar{x}_1} \left[\overline{\sigma}\,(b + \psi\tan\theta) + \bar{c}\tan\theta\right] d\bar{x}$$

$$- \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right) d\bar{x} \text{------------------------} 3.56$$

$$\overline{Q} - \overline{H}\cos\beta = \int_{x_0}^{\bar{x}_1} \left[\overline{\sigma}\left(1 + \psi\,\bar{y}^{-1}\right) + \bar{c}\,\bar{y}^{-1}\right] dx - \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right) d\bar{x} \text{----------} 3.57$$

Now $\cos\beta = H/B$ fig. 3.1 and fig. 3.2 and from equation 3.37 $\overline{H} = H/B$

$$\therefore \cos\beta = \overline{H} \text{----------------------} 3.58$$

Substitute equation 3.58 into 3.57 thus

$$\overline{Q} - \overline{H}^2 = \int_{x_0}^{\bar{x}_1} \left[\overline{\sigma}\left(1 + \psi\,\bar{y}^{-1}\right) + \bar{c}\,\bar{y}^{-1}\right] dx - \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right) dx \text{----------} 3.59$$

Now, further treatment of equation 3.39, 3.40 and 3.41 results in the following:

$$\bar{c} = \hat{c} - \overline{H} \text{----------------------} 3.60$$

$$\overline{\sigma} = \hat{\sigma} + \overline{H} \text{---------------------} 3.61$$

$$\overline{Q} = \hat{Q} + 2\overline{H} \text{--------------------} 3.62$$

Substituting therefore equations 3.60, 3.61 and 3.62 into equation 3.59, we have

$$\underset{-\overline{H}^2}{\bar{Q}} = \int_{x_0}^{\bar{x}_1} \left[\left(\hat{\sigma} + \overline{H}\right)\left(1 + \psi\,\bar{y}^{-1}\right) + \left(\hat{c} - \psi H\right)\bar{y}^{-1}\right] dx - \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right) d\bar{x} \text{----} 3.63$$

Further expanding, we get

$$\overline{Q} - \overline{H}^2 = \int_{x_0}^{\bar{x}_1} \left[\hat{\sigma}\left(1 + \psi\,\bar{y}^1\right) + \hat{c}\,\bar{y}^1 + \overline{H}\right] d\bar{x} - \int_{x_0}^{\bar{x}_1} \left(\bar{y} + \overline{H}\right) d\bar{x} \text{----} 3.64$$

$$= \int_{x_0}^{\bar{x}_1} \left[\hat{\sigma}\left(1 + \psi\,\bar{y}^1\right) + \hat{c}\,\bar{y}^1 + \overline{H} - \bar{y} - \overline{H}\right] dx \text{--------------} 3.65$$

$$= \int_{\bar{x}_0}^{\bar{x}_1} \left[\hat{\sigma}\left(1+\psi\,\bar{y}^1\right) + \hat{c}\,\bar{y} - \bar{y}\right] dx \text{ ---------------------- } 3.66$$

$$= \int_{\bar{x}_0}^{\bar{x}_1} \left[\hat{\sigma}\left(H+\psi\,\bar{y}^1\right) - \bar{y}\right] d\bar{x} + \int_{\bar{x}_0}^{\bar{x}_1} \hat{c}\,\bar{y}^1 \, d\bar{x} \text{ ------------- } 3.67$$

$$= \int_{\bar{x}_0}^{\bar{x}_1} \left[\hat{\sigma}\left(1+\psi\,\bar{y}^1\right) - \bar{y}\right] d\bar{x} + \hat{c}\int_{\bar{x}_0}^{\bar{x}_1} \bar{y}^1 \, d\bar{x} \text{ -------------- } 3.68$$

By invoking the result of equation 3.28, equation 3.68 becomes

$$\bar{Q} - \bar{H}^2 = \int_{\bar{x}_0}^{\bar{x}_1} \left[\sigma\left(1+\psi\,\bar{y}^1\right) - \bar{y}\right] d\bar{x} \text{ -------------------- } 3.69$$

(b) For equation 3.16, we have

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\sigma\left(\sin\theta - \psi\cos\theta\right) - c\cos\theta\right] ds = 0$$

Introducing the non-dimensional parameters of equations 3.44 and 3.46, we obtain

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\bar{\boldsymbol{\sigma}}\,\gamma B\left(\sin\theta - \psi\cos\theta\right) - \bar{c}\,\gamma B\cos\theta\right] \frac{Bdx}{\cos\theta} = 0 \text{ ------ } 3.70$$

$$\gamma B^2 \int_{\bar{x}_0}^{\bar{x}_1} \left[\bar{\sigma}\left(\sin\theta - \psi\cos\theta\right) - \bar{c}\cos\theta\right] \frac{d\bar{x}}{\cos\theta} = 0 \text{ ----------- } 3.71$$

$$\gamma B^2 \int_{\bar{x}_0}^{\bar{x}_1} \left[\bar{\sigma}\left(\tan\theta - \psi\right) - \bar{c}\right] d\bar{x} = 0 \text{ ------------------ } 3.72$$

Dividing although by γB^2, we get

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\bar{\sigma}\left(\tan\theta - \psi\right) - \bar{c}\right] d\bar{x} = 0 \text{ ------------------ } 3.73$$

Introducing equations 3.60 and 3.61 into 3.73, we get

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\left(\bar{\sigma} + \bar{H}\right) \left(\tan\theta - \psi\right) - \left(\hat{c} - \psi \bar{H}\right) \right] d\bar{x} = 0 \text{ ---------------- } 3.74$$

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\left(\bar{\sigma} + \bar{H}\right) \left(\bar{y}^1 - \psi\right) - \left(\hat{c} - \psi \bar{H}\right) \right] d\bar{x} = 0 \text{ ------------------ } 3.75$$

$$\int_{\bar{x}_0}^{\bar{x}_1} \left[\bar{\sigma}\, \bar{y}^1 - \bar{\sigma}\psi + \bar{H}\, \bar{y}^1 - \bar{H}\psi - \hat{c} + \psi \bar{H} \right] d\bar{x} = 0 \text{ ------------ } 3.76$$

$$\int_{x_0}^{x_1} \left[\bar{\sigma} \left(\bar{y}^1 - \psi\right) + \bar{H}\, \bar{y}^1 - \hat{c} \right] dx = 0 \text{ ------------------- } 3.77$$

$$\int_{x_0}^{\bar{x}_1} \left[\bar{\sigma} \left(\bar{y}^1 - \psi\right) - \hat{c} \right] dx + \int_{x_0}^{\bar{x}_1} \bar{H}\, \bar{y}^1 \, dx = 0 \text{ ------------- } 3.78$$

$$\int_{x_0}^{\bar{x}_1} \left[\bar{\sigma} \left(\bar{y}^1 - \psi\right) - \hat{c} \right] d\bar{x} + \bar{H} \int_{x_0}^{\bar{x}_1} \bar{y}^1 \, d\bar{x} = 0 \text{ ------------ } 3.79$$

Invoking the result of equation 3.28, the equation 3.79 simplifies to

$$\int_{x_0}^{\bar{x}_1} \left[\bar{\sigma} \left(\bar{y}^1 - \psi\right) - \hat{c} \right] d\bar{x} = 0 \text{ -------------------- } 3.80$$

(c) For equation 3.20, we have

$$\int_s \left\{ \left[\sigma \left(\psi \cos\theta - \sin\theta\right) + c\, \mathrm{Cos}\,\theta \right] y - \left[\sigma \left(\cos\theta + \psi \sin\theta\right) + c \sin\theta \right] x \right.$$

$$\left. ds + \int_{x_0}^{\bar{x}_1} ४\, x \left(y + H\right) dx = 0 \right\}$$

Introducing the non-dimensional parameters of equations 3.44 and 3.46 into 3.20 gives us:

$$४ B^2 \int_{\bar{x}_0}^{\bar{x}_1} \left\{ \left[\bar{\sigma} \left(\psi - \tan\theta\right) + \bar{c} \right] \bar{y} - \left[\bar{\sigma} \left(1 + \psi \tan\theta\right) + \bar{c} \tan\theta \right] \bar{x} \right\} d\bar{x} +$$

$$४ B^2 \int_{x_0}^{\bar{x}_1} \bar{x} \left(\bar{y} + \bar{H}\right) d\bar{x} = 0 \text{ -------------- } 3.81$$

By using equations 3.43 and 3.44 and making necessary substitutions;

$$४ B^2 \int_{x_0}^{\bar{x}_1} \left\{ \left[\bar{\boldsymbol{\sigma}} \left(\boldsymbol{\psi} - \bar{y}^{-1}\right) + \bar{c} \right] \bar{y} - \left[\bar{\boldsymbol{\sigma}} \left(1 + \boldsymbol{\psi}\, \bar{y}^{-1}\right) + \bar{c}\, \bar{y}^{-1} \right] \bar{x} \right\} B d\bar{x}$$

$$+ \gamma B^2 \int_{x_0}^{\bar{x}_1} \bar{x}\left(\bar{y} + \overline{H}\right) B d\bar{x} = 0 \text{ ----------- } 3.82$$

Dividing although by rB^2, we have

$$\int_{x_0}^{\bar{x}_1} \left\{\left[\bar{\sigma}\left(\psi - \bar{y}^{1}\right) + \bar{c}\right]\bar{y} - \left[\bar{\sigma}\left(1 + \psi \bar{y}^{1}\right) + \bar{c}\,\bar{y}^{1}\right]\bar{x}\right\} dx$$

$$+ \int_{x_0}^{\bar{x}_1} \bar{x}\left(\bar{y} + \overline{H}\right) d\bar{x} = 0 \text{ ----------------------- } 3.83$$

Introducing the results of equation 3.46 into equation 3.83 gives

$$\int_{x_0}^{\bar{x}_1} \left\{\left[\left(\hat{\sigma} + \overline{H}\right)\left(\psi - \bar{y}^{1}\right) + \hat{c} - \psi \overline{H}\right]\bar{y} - \left[\left(\hat{\sigma} + \overline{H}\right)\left(1 + \psi \bar{y}^{1}\right) + \left(\hat{c} - \psi \overline{H}\right)\bar{y}^{1}\right] x\right\}$$

$$d\bar{x} + \int_{x_0}^{\bar{x}_1} \bar{x}\left(\bar{y}' + \vec{H}\right) d\bar{x} = 0 \text{ ---------- } 3.84$$

Simplifying

$$\int_{x_0}^{\bar{x}_1} \left\{\left[\hat{\sigma}\psi\bar{y} + \hat{\sigma}\psi\bar{y}\,\bar{y}^1 - \overline{H}\,\bar{y}\,\bar{y}^1 + \hat{c}\bar{y}\right] - \left[\hat{\sigma}\bar{x} + \hat{\sigma}\psi\bar{x}\,\bar{y}^1 + \bar{y}\,\bar{x} + \hat{c}\,\bar{y}^1\bar{x}\right]\right\} d\bar{x}$$

$$+ \int_{x_0}^{\bar{x}_1} \left(\bar{x}\,\bar{y} + \overline{H}\,\bar{x}\right) d\bar{x} = 0 \text{ ------------------ } 3.85$$

Taking like terms together

$$\int_{x_0}^{\bar{x}_1} \left\{\hat{\sigma}\left(\psi\bar{y} - \psi\bar{x}\,\bar{y}^1 - \bar{x} - \bar{y}\,\bar{y}^1\right) + \hat{c}\left(\hat{y} - \bar{x}\,\bar{y}\right) - \overline{H}\bar{y}\,\bar{y} - \overline{H}\bar{x} + \overline{H}\bar{x} + \bar{x}\bar{y}\right\} d\bar{x} = 0 \text{ ---- } 3.86$$

$$\int_{x_0}^{\bar{x}_1} \left\{\hat{\sigma}\left[\psi\left(\bar{y} - \bar{x}\,\bar{y}^1\right) - \left(\bar{x} + \bar{y}\,\bar{y}^1\right)\right] + \hat{c}\left(\bar{y} + \bar{x}\,\bar{y}^1\right) + \bar{y}\left(\bar{x} - \overline{H}\,\bar{y}^1\right)\right\} d\bar{x} = 0 \text{ -- } 3.87$$

But the result of equation 3.33

$$\int_{x_0}^{\bar{x}_1} \bar{y}\,\bar{y}^1 d\bar{x} = 0$$

and so equation 3.56 becomes

$$\int_{x_0}^{\bar{x}_1} \left\{\hat{\sigma}\left[\psi\left(\bar{y} - \bar{x}\bar{y}^1\right) - \left(\bar{x} + \bar{y}\,\bar{y}^1\right)\right] + \hat{c}\left(\bar{y} - \bar{x}\,\bar{y}^1\right) + \bar{x}\,\bar{y}\right\} d\bar{x} = 0 \text{ ---- } 3.88$$

The basic five parameters of the problem (c, ϕ, ɤ, H, B) enter into the system of equations represented by 3.69, 3.80 and 3.88 in the combination of ψ and c only. Thus the transformation into non-dimensional parameters has effectively and advantageously reduced the number of problem parameters from five to two.

Construction of Euler-Lagrangian Intermediate Function for the Problem

Consider the stability function of equation 3.69 given as

$$\bar{Q} - \bar{H}^2 = \int_{x_0}^{\bar{x}_1} \left[\hat{\sigma}\left(1 + \psi\,\bar{y}^1\right) - \bar{y}\right] dx ------------------3.89$$

Denote the integrand by U which is now a function of σ, y, y^1, and c. This implies that

$$U\left(\sigma, \bar{y}, \bar{y}^1, \psi, C\right) = \hat{\sigma}\left(\psi\,\bar{y}^1 + 1\right) - \bar{y} ----------3.90$$ and

$$\bar{Q} - \bar{H}^2 = \int_{x_0}^{\bar{x}_1} U\left(\hat{\sigma}, \bar{y}, \bar{y}^1, \psi, \hat{c}\right) d\bar{x} ------------------3.91$$

$$(but\ \bar{H} = \cos\alpha \ \therefore\ \bar{H}^2 = \cos^2\alpha)$$

For the stability function of equation 3.80 representing the horizontal equilibrium state, the equation is

$$\int_{x_0}^{\bar{x}_1} \left[\hat{\sigma}\left(\bar{y}^1 - \psi\right) - \hat{c}\right] d\bar{x} = 0$$

the integrand is denoted by V, then V which is now a function of σ, y^1, ψ, and c becomes

$$V\left(\bar{\sigma}, \bar{y}^1, \psi, \hat{c}\right) = \bar{\sigma}\left(\bar{y}^1 - \psi\right) - \hat{c} ------------------3.92$$

$$and \int_{x_0}^{\bar{x}_1} V\left(\bar{\sigma}, \bar{y}^1, \psi, \hat{c}\right) d\bar{x} = 0 ------------------3.93$$

Similarly, denote by W, the integrand of the stability function representing the moment equilibrium and given in equation 3.88 as

$$\int_{x_0}^{\bar{x}_1} \left\{\hat{\sigma}\left[\psi\left(\bar{y} - \bar{x}\,\bar{y}^1\right) - \left(\bar{x} + \bar{y}\,\bar{y}^1\right)\right] + \hat{c}\left(\bar{y} - \bar{x}\,\bar{y}^1\right) + \bar{x}\,\bar{y}\right\} d\bar{x} = 0$$

Obviously W is a function of σ, $\bar{y}$, $\bar{y}^1$, ψ, $\hat{c}$ *and* $\bar{x}$ and so

$$W\left(\hat{\sigma},\ \bar{y},\ \bar{y}^1,\ \psi,\ \bar{x},\ \hat{c}\right) = \left\{\hat{\sigma}\ \left[\psi\ \left(\bar{y} - \bar{x}\,\bar{y}^1\right) - \left(\bar{x} + \bar{y}\,\bar{y}^1\right)\right] + \hat{c}\left(\bar{y} - \bar{x}\,\bar{y}^1\right) + \bar{x}\,\bar{y}\right\} --- 3.94$$

$$\int_{x_0}^{\bar{x}_1} W\left(\hat{\sigma},\ \bar{y},\ \bar{y}^1,\ \psi,\ \bar{x},\ \hat{c}\right) = 0 \ ------------------ 3.95$$

The solution of the foregoing variational problem will now be constructed using the method of Lagrange's immediate multipliers. In line with this is defined an intermediate function S (Swokowski, 1991; Elsgolts, 1977).

$$S = U + \lambda_1 V + \lambda_2 W \ ---------------------- 3.96$$

which is seen to incorporate the load function U and the necessary constraints V and W. $\lambda_1, \lambda_2,$ are Lagrange's undetermined multipliers. Replacing U, V, and W with their appropriate expressions from equations 3.90, 3.92 and 3.94, we have

$$\begin{aligned} S = {} & \left[\hat{\sigma}\ \left(\psi\bar{y}^1 + 1\right) - \bar{y}\right] + \lambda_1\ \left[\hat{\sigma}\ \left(\bar{y}^1 - \psi\right) - \hat{c}\right] \\ & + \lambda_2\ \left\{\hat{\sigma}\ \left[\psi\ \left(\bar{y} - \bar{x}\,\bar{y}^1\right) - \bar{x} + \bar{y}\,\bar{y}^1\right] + \hat{c}\ \left(\bar{y} - \bar{x}\,\bar{y}^1\right) + \bar{x}\,\bar{y}\right\} ------- 3.97 \end{aligned}$$

The equation 3.97 for s integrates the load (objective) function with the constraints. It is the functional which itself is a function of two functions $\bar{y}(x)$, the rupture surface and $\hat{\sigma}(x)$, the normal stress distribution on the rupture surface.

In the subsequent section, S is immunized with respect to the functions $\bar{y}(x)$ and $\hat{\sigma}(x)$ by subjection to the appropriately constructed Euler-Lagrange's differential equation. The determination of the expressions for the critical normal stress distribution $\hat{\sigma}(x)$ and the critical rupture surface $\bar{y}(x)$ and which ultimately results from the minimization of S thus the main thrust of the bearing capacity problem.

Formulation of Euler-Lagrange Differential Equation

The criticals $\hat{\sigma}(x)$ and $\bar{y}(x)$ must necessarily satisfy

(a) system of Euler differential equation in S

(b) the integral constraints of equations 3.80 and 3.88

(c) the set of boundary conditions at the end points $\bar{x}_0$ *and* $\bar{x}_1$

For the differential equation, Euler had theorized that for a functional of one function $[\bar{y}\ (\bar{x})]$

$$J(\bar{y}) = \int_{x_0}^{\bar{x}_1} F\ \left(y,\ y^1,\ y'',\ ...,\ y^n\right)dx -------------- 3.98$$

The appropriate differential equation is (Swokowski, 1991; Elsgolts, 1977)

$$Fy - \frac{d}{dx} Fy^1 + \frac{d^2}{dx^2} Fy'' ----+ (-1)^n \frac{d^n}{dx^n} Fy^n = 0 ---------- 3.99$$

$$\left.\begin{aligned} \text{when } Fy &= \frac{\partial F}{\partial y} \\ Fy^1 &= \frac{\partial F}{\partial y^1} \end{aligned}\right\} ------------------------ 3.100$$

$$Fy^n = \frac{\partial F}{\partial y^n}$$

Thus the Euler differential equation of equation 3.99 becomes

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y^1}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) --- + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0 -------- 3.101$$

In the same light, for a functional of two functions y(x) and z(x),

$$J\ \left[y(x),\ (x)\right] = \int_{x_0}^{\bar{x}_1} F\ \left(x,\ y,\ z,\ y^1,\ z^1,\ y'',\ z'',\ ----,\ y^n,\ z^n\right) dx ---- 3.102$$

and the system of Euler differential equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y^1}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) --- + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z^1}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial z''}\right) --- + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0 ------ 3.103$$

For the particular case of the formulated problem, we have that the functional S is a function of two variables incorporating first order differential. The appropriate Euler's differential equation for the present problem may therefore be written as

$$\frac{\partial s}{\partial \hat{\sigma}} - \frac{d}{dx}\left(\frac{\partial s}{\partial \hat{\sigma}^1}\right) = 0 --------------------- 3.104$$

$$\frac{\partial s}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial s}{\partial \bar{y}^1}\right) = 0 --------------------- 3.105$$

further, bringing the condition of transversality, i.e., the variational boundary condition of equation 3.34

$$S - \bar{y}^1 \frac{\partial s}{\partial y^1} - \hat{\sigma}^1 \frac{\partial s}{\partial \hat{\sigma}^1} \Bigg|_{\bar{x} = \bar{x}_1} = 0$$

Now since S in equation 3.97 does not depend on $\hat{\sigma}$, then equation S 3.104, 3.105 and 3.34 simplify to

$$\frac{\partial s}{\partial \hat{\sigma}} = 0 \text{ ----------------------- } 3.106$$

$$\frac{\partial s}{\partial \bar{y}} - \frac{d}{d\bar{x}} \left(\frac{\partial s}{\partial \bar{y}^1} \right) = 0 \text{ -------------- } 3.107$$

$$\left(S - \bar{y}^1 \frac{\partial s}{\partial y^1} \right) \Bigg|_{\bar{x} = \bar{x}_1} = 0 \text{ ----------- } 3.108$$

Thus the problem reduces to that of solving the two differential equations, Equations 3.106 and 3.107, subject to the fulfillment of the two integral constraints Equation 3.80 and 3.88, the geometrical boundary conditions, equations 3.22 and 3.23 and transversality condition equation 3.108.

Co-ordinate Transformation and General Solution

Co-ordinate Transformation

From equation 3.97, we discover that S is linear in σ, and so equation 3.106 is independent in σ, and is a first order differential equation in y only. It is solved independent of Euler's second equation 3.107. The solution which is found elsewhere (Ike, 1979; Michael and Raphael, 1977) results into an expression for the critical rupture surface which is found to be logarithmic spiral curve.

Following a rigorous process and using polar coordinate system, the expression for the critical normal stress distribution σ(x) is obtained by a complete solution of equation 3.103. It is found convenient to introduce the following coordinate transformation.

$$x = r\cos\theta + \frac{1}{\lambda_2} \text{ ---------------------- } 3.109$$

$$y = r\sin\theta + \frac{\lambda_1}{\lambda_2} \text{ ---------------------- } 3.110$$

when (r, θ) is a polar coordinate system centered around the point (x_r, y_r):

$$\bar{x}_r = \frac{1}{\lambda_2} \text{ ------------------------- } 3.111$$

$$\bar{y}_r = \frac{\lambda_1}{\lambda_2} \text{ ----------------------- } 3.112$$

$$But \quad \bar{y}_1 = \frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}}{d\theta} * \frac{d\theta}{d\bar{x}} \quad (Chain\ rule\ of\ differention)\ [29]$$

$$= \frac{d\bar{y}}{d\theta} * \frac{1}{\frac{d\bar{x}}{d\theta}} \text{ --------------------------------- } 3.113$$

Now y (eq 3.110) is a function of two variables $\bar{r}$ and θ. So we use product rule thus:

$$\frac{d}{dx}\{U(x), V(x)\} = U(x)\frac{dV(x)}{dx} + V(x)\frac{du(x)}{dx} \text{ ----------- } 3.114$$

so that

$$\frac{d\bar{y}}{d\theta} = \bar{r}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d\bar{r}}{d\theta}$$

$$= \bar{r}\cos\theta + \sin\theta\frac{d\bar{r}}{d\theta} \text{ -------------------- } 3.115$$

similarly

$$\frac{d\bar{x}}{d\theta} = r\frac{d}{d\theta}(\cos\theta) + \cos\theta\frac{d\bar{r}}{d\theta} = -\bar{r}\sin\theta + \cos\theta\frac{d\bar{r}}{d\theta} \text{ ------ } 3.116$$

(Note that λ_1 and λ_2 are constants and so result to zero on differentiation).

Thus, by introducing the results of equations 3.115 and 3.116 into equation 3.113,

$$\bar{y}^1 = \frac{d\bar{y}}{d\theta} * \frac{1}{\frac{d\bar{x}}{d\theta}}$$

we have

$$\bar{y}^1 = \bar{r}\cos\theta + \sin\theta\frac{d\bar{r}}{d\theta} + \frac{1}{Cos\ \theta\frac{d\bar{r}}{d\theta} - \bar{r}\sin\theta}$$

$$= \frac{\bar{r}\cos\theta + \sin\theta \dfrac{d\bar{r}}{d\theta}}{\cos\dfrac{d\bar{r}}{d\theta} - \bar{r}\sin\theta} \quad ------------------- 3.117$$

similarly

$$\sigma^1 = \frac{d\sigma}{d\theta} * \frac{d\theta}{dx} \quad [29] \quad = \frac{d\sigma}{d\theta} * \frac{1}{\dfrac{d\bar{x}}{d\theta}} \quad ------------- 3.118$$

and introducing equation 3.116 into 3.118 takes us to

$$\sigma^1 = \frac{d\sigma}{d\theta} * \frac{1}{\left(\cos\theta \dfrac{d\bar{r}}{d\theta} - \bar{r}\sin\theta\right)} \quad -------------- 3.119$$

The solution of the Euler first differential equation has already been dealt with. The result was obtained by introducing the definition of S (equation 3.97) into the Eulers first differential equation, Equation 3.106 and using the coordinate transformation equations 3.109, 3.110 and 3.117. A resulting first order differential equation

$$\frac{1}{r}\frac{d\bar{r}}{d\theta} = -\psi \quad ----------------- 3.120$$

was solved for r to obtain

$$\bar{r}(\theta) = \bar{r}_0 . e^{(\theta_0 - \theta)^{\psi}} \quad -------------- 3.121$$

in which (r_o, θ_o) are the constants of integration that may be conveniently taken as polar coordinate of point (r_o, y_o).

The above equation, equation 3.121 is identified as the equation of a logarithmic spiral curve and which is the shape of the critical rupture surface. To solve the Euler second equation to obtain the normal stress distribution $\hat{\sigma}(\bar{x})$ on the critical rupture surface, we introduce the definition of S in the Euler's second equation, Equation 3.107, thus:

Recall

$$S\left[\hat{\sigma}\left(\psi\,\bar{y}^1+1\right)-\bar{y}\right]+\lambda_1\left[\hat{\sigma}\left(\bar{y}^1-\psi\right)-\hat{c}\right]$$
$$+\lambda_2\left\{\hat{\sigma}\left(\psi\,\bar{y}^1-\bar{x}\,\bar{y}^1\right)-\left(\bar{x}+\bar{y}\,\bar{y}^1\right)+\hat{c}\left(\bar{y}-\bar{x}\,\bar{y}^1\right)+\bar{x}\,\bar{y}\right\}----$$
$$\frac{\partial s}{\partial y}=-1+\lambda_2\left[\hat{\sigma}\psi-\hat{\sigma}\bar{y}^1+\hat{c}+\bar{x}\right]-------------------------3.122$$
$$=-1+\lambda_2\hat{\sigma}\psi-\lambda_2\hat{\sigma}\,\bar{y}^1+\lambda_2\hat{c}+\lambda_2\bar{x}-----------------------3.123$$
$$\frac{\partial s}{\partial y^1}=\hat{\sigma}\psi+\lambda_1\hat{\sigma}+\lambda_2\left\{\hat{\sigma}\,(-x)\,\psi-\hat{\sigma}\,\bar{y}-\hat{c}\,\bar{x}\right\}-----------------3.124$$
$$=\hat{\sigma}\left(\psi+\lambda_1-\lambda_2\,\bar{x}\psi-\lambda_2\,\bar{y}\right)-\lambda_2\,\hat{c}\,\bar{x}--------------------3.125$$
$$\frac{d}{dx}\left(\frac{\partial s}{\partial y^1}\right)=\hat{\sigma}^1\left(\psi+\lambda_1-\lambda_2\,\bar{x}\psi-\lambda_2\,\bar{y}\right)+\hat{\sigma}\left(\lambda_2\psi-\lambda_2\bar{y}^1\right)-\lambda_2\hat{c}---3.126$$

But Euler's second equation is recalled thus

$$\frac{\partial s}{\partial \bar{y}}-\frac{d}{dx}\left(\frac{\partial s}{\partial \bar{y}^1}\right)=0$$

Substituting equations 3.123 and 3.126 into 3.107 results into:

$$-1+\lambda_2\hat{\sigma}\psi-\lambda_2\hat{\sigma}\,\bar{y}^1+\lambda_2\hat{c}+\lambda_2\bar{x}-\hat{\sigma}^1\left(\psi+\lambda_1-\lambda_2\;x\,\psi-\lambda_2\bar{y}\right)$$
$$+\lambda_2\hat{\sigma}\psi+\lambda_2\hat{\sigma}\,\bar{y}^1+\lambda_2\hat{c}=0----------------------3.127$$

Simplifying

$$-1+2\lambda_2\hat{\sigma}\psi+2\lambda_2\hat{c}+\lambda_2 x-\hat{\sigma}^1\left(\psi+\lambda_1-\lambda_2\;x\psi-\lambda_2\bar{y}\right)=0---------3.128$$

Transforming polar-coordinate – wise by introducing the expressions for y and x from equations 3.109 and 3.110, we have

$$-1+2\lambda_2\hat{\sigma}\psi+2\lambda_2\hat{c}+\lambda_2\left(r\cos\theta+\frac{1}{\lambda_2}\right)$$
$$-\sigma^1\left[\psi+\lambda_1-\lambda_2\psi\left(r\cos\theta+\lambda_2\right)-\lambda_2\left(r\sin\theta+\frac{1}{\lambda_2}\right)\right]=0---------3.129$$

$$-1+2\lambda_2\sigma\psi+2\lambda_2\hat{c}+\lambda_2\,r\cos\theta+1-\hat{\sigma}^1$$
$$\left(\psi+\lambda_1-\lambda_2\psi\;r\cos\theta-\psi-\lambda_2\,r\sin\theta-\lambda_1\right)=0--------------3.130$$

$$-1+2\lambda_2\hat{\sigma}\psi + 2\lambda_2\hat{c} + \lambda_2 r\cos\theta + 1-\hat{\sigma}^1 \left(-\lambda_2\psi\, r\cos\theta - \lambda_2 r\sin\theta\right) = 0 ---3.131$$

$$-1 + 2\lambda_2\hat{\sigma}\psi + 2\lambda_2\hat{c} + \lambda_2 r\cos\theta + 1 + \lambda_2\hat{\sigma}^1 \left(\psi\, r\cos\theta + r\sin\theta\right) = 0 ---3.132$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \hat{\sigma}^1 r \left(\psi\cos\theta + \sin\theta\right) = 0 -------------3.133$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\sigma}{dx}\cdot r \left(\psi\cos\theta + \sin\theta\right) = 0 ---------3.134$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\sigma}{d\theta}\cdot\frac{1}{\frac{dx}{d\theta}}\cdot r\left(\psi\cos\theta + \sin\theta\right) = 0 ------3.135$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta \frac{d\hat{\sigma}}{d\theta}\cdot\frac{1}{\frac{dx}{d\theta}}\cdot r\left(\psi\cos\theta + \sin\theta\right) = 0 -------3.136$$

Introducing the expression for $\frac{dx}{d\theta}$ from equation 3.116 into equation 3.136, we obtain

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\hat{\sigma}}{d\theta}\;\frac{1}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)}\; r\left(\psi\cos\theta + \sin\theta\right) = 0 ---3.137$$

From equation 3.120, we see that

$$\frac{dr}{d\theta} = -r\psi -----------------------3.138$$

Introduce equation 3.138 into equation 3.137 thus

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\hat{\sigma}}{d\theta}\;\frac{r\left(\psi\cos\theta + \sin\theta\right)}{\left[\cos\theta\left(-r\psi\right) - r\sin\theta\right]} = 0 -------3.139$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\hat{\sigma}}{d\theta}\cdot\frac{r\left(\psi\cos\theta + \sin\theta\right)}{-r\left(\psi\cos\theta + \sin\theta\right)} = 0 --------3.140$$

$$2\hat{\sigma}\psi + 2\hat{c} + r\cos\theta + \frac{d\hat{\sigma}}{d\theta} = 0 ---------------------3.141$$

substituting the expression for r(θ), equation 3.121 into 3.141 results to

$$2\hat{\sigma}\psi + 2\hat{c} + r_0 e^{[(\theta_0-\theta)\psi]}\cos\theta - \frac{d\hat{\sigma}}{d\theta} = 0 -------------------3.142$$

$$\frac{d\hat{\sigma}}{d\theta} - 2\hat{\sigma}\psi - 2\hat{c} - r_0 e^{[(\theta_0-\theta)\psi]}\cos\theta = 0 --------------------3.143$$

Rearranging equation 3.143, thus

$$\frac{d\hat{\sigma}}{d\theta} - 2\hat{\sigma}\psi - \{r_0 \exp[(\theta_0 - \theta)\psi] \cos\theta + 2\hat{c}\} = 0 \text{-------------}3.144$$

We can clearly see that it is a first order linear differential equation in $\hat{\sigma}$. This is solved by procedure of separation of variables.

Solution of the Resulting Differential Equation

If we rearrange the differential equation, equation 3.144, we get

$$\frac{d\hat{\sigma}}{d\theta} - 2\hat{\sigma}\psi = r_0 e^{[(\theta_0 - \theta)\psi]} \cos\theta + 2\hat{c} \text{--------------------}3.145$$

Equation 3.145 is a first order linear non-homogeneous differential equation and which we solve by separating the variable thus [29]

$$Let \ f(\theta) = -2\psi \text{------------------------------}3.146$$

$$h = \int f(\theta)\, d\theta = -2\psi\theta \text{-------------------------}3.147$$

$$g = r_0 e^{[(\theta_0 - \theta)\psi]} \cos\theta + 2c \text{-----------------------}3.148$$

Then

$$\hat{\sigma}(\theta) = e^{-h}\left[\int e^h . g . d\theta + B\right] \text{--------------------}3.149$$

substituting Equations 3.146, 3.147 and 3.148 into equation 3.149 takes us:

$$\hat{\sigma}(\theta) = e^{2\psi\theta}\left[\int e^{-2\psi\theta}\{r_0 e^{[(\theta_0 - \theta)\psi]} \cos\theta + 2c\} d\theta + B\right] \text{--------}3.150$$

$$= e^{-2\psi\theta}\left[\int r_0 e^{\psi\theta_0} e^{-3\psi\theta} \cos\theta)\, d\theta + \int 2ce^{-2\psi\theta}\, d\theta + B\right] \text{------}3.151$$

$$= e^{2\psi\theta}\left[r_0 e^{\psi\theta_0} \int e^{-3\psi\theta} \cos\theta\, d\theta + 2c\int e^{-2\psi\theta}\, d\theta + B\right] \text{-----}3.152$$

$$= r_0 e^{(2\theta + \theta_0)\psi} \int e^{-3\psi\theta} \cos\theta\, d\theta + 2ce^{2\psi\theta}\int e^{-2\psi\theta}\, d\theta + Be^{2\psi\theta} \text{----}3.153$$

$$= r_0 e^{(\theta_0 + 2\theta)\psi} \int e^{-3\psi\theta} \cos\theta\, d\theta - \frac{c}{\psi} e^{(2\psi\theta - 2\psi\theta)} + Be^{2\psi\theta} \text{-------}3.154$$

$$= r_0 e^{(\theta_0 + \theta)\psi} \int e^{-3\psi\theta} \cos\theta\, d\theta - \frac{c}{\psi} + Be^{2\psi\theta} \text{------------}3.155$$

$$But \int e^{-3\psi\theta} \cos\theta\, d\theta = \frac{e^{-3\psi\theta}}{1 + 9\psi^2}(\sin\theta - 3\psi\cos\theta) \text{------------}3.156$$

(see Appendix A)

substituting equation 3.156 into 3.155 gives

$$\hat{\sigma}(\theta) = r_0 e^{\psi(\theta_0 + 2\theta)} \left[\frac{e^{-3\psi\theta}}{1 + 9\psi^2} \left(\sin\theta - 3\psi\cos\theta\right)\right] - \frac{c}{\psi} + Be^{2\psi\theta} ------3.157$$

$$= r_0 \frac{e^{\psi(\theta_0 - \theta)}}{1 + 9\psi^2} \left(\left(\sin\theta - 3\psi\cos\theta\right) - \frac{c}{\psi} + Be^{2\psi\theta} \right) --------3.158$$

codifying and rearranging, we get

$$\sigma(\theta) = r_0 A(\theta) + Be^{2\psi\theta} - \frac{c}{\psi} -------------------3.159$$

$$where\ A(\theta) = \frac{e^{(\theta_0 - \theta)}}{1 + 9\psi^2} \left(\sin\theta - 3\psi\cos\theta\right) ----------------3.160$$

B = integrating constant.

Now for a case where ψ = tan ϕ = 0; i.e., frictionless soil, we substitute this zero value into equation 3.153 before performing the integration thus:

$$\sigma(\theta) \quad r_0 e^{\psi(\theta_0 + 2\theta)} \int e^{-3\psi\theta} \cos\theta\, d\theta + 2ce^{2\psi\theta} \int e^{-2\psi\theta}\, d\theta + Be^{2\psi\theta} ----$$

$$substitute \quad \psi = 0$$

$$\sigma(\theta) = r_0 \int \cos\theta\, d\theta + 2c \int d\theta + B ----------------------3.161$$

$$= r_0 \sin\theta + 2c\theta + B = 0 -------------------------3.162$$

Solution of Transversality condition (Variational Boundary Condition)

The expression of the variational boundary condition is given in equation 3.108 as

$$S - y^1 \frac{\partial s}{\partial y^1} \Bigg|_{x_1 = x_1} = 0$$

Applying the definition of S of equation 3.97 here,

$$S = \left[\hat{\sigma}\left(\psi\, \bar{y}^1 + 1\right) - \bar{y}\right] + \lambda_1 \left[\hat{\sigma}\left(\bar{y}^1 - \psi\right) - \hat{c}\right]$$
$$+ \lambda_2 \left\{\hat{\sigma}\left[\psi\left(\bar{y} - \bar{x}\,\bar{y}^1\right) - \left(\bar{x} + \bar{y}\,\bar{y}^1\right)\right] + \hat{c}\left(\bar{y} - \bar{x}\,\bar{y}\right) + \bar{x}\,\bar{y}\right\} ----$$

$$\frac{\partial s}{\partial y^1} = \hat{\sigma}\psi + \lambda_1\hat{\sigma} + \lambda_2\psi\hat{\sigma}\ (-\bar{x}) + \lambda_2\hat{\sigma}\ (-\bar{y}) + \hat{c}\ (-\bar{x})\ \lambda_2$$

$$= \hat{\sigma}\psi + \lambda_1\hat{\sigma} - \lambda_2\ \{\hat{\sigma}\,\bar{x}\,\psi + \hat{\sigma}\,\bar{y} + \hat{c}\,\bar{x}\} ---------------------- 3.163$$

$$\bar{y}^1\ \frac{\partial s}{\partial y^1} = \hat{\sigma}\psi\,\bar{y}^1 + \lambda_1\hat{\sigma}\,\bar{y}^1 - \lambda_1\,\bar{y}^1\ (\hat{\sigma}\,\bar{x}\,\psi + \hat{\sigma}\,\bar{y} + \hat{c}\,\bar{x}) ------------- 3.164$$

$$S - \bar{y}^1\ \frac{\partial s}{\partial y^1} = 0 \quad (\text{Re}\,call) -------------------------$$

This implies, further necessary substitution with equations 3.97 and 3.164 and expanding

$$\hat{\sigma}\psi\,y^1 + \hat{\sigma} - \bar{y} + \lambda_1\hat{\sigma}\,\bar{y}^1 - \lambda_1\hat{\sigma}\psi - \lambda_1\hat{c} + \lambda_2\psi\hat{\sigma}y -$$
$$\lambda_2\hat{\sigma}\psi\,\bar{x}\,y^1 - \lambda_2\hat{\sigma}\,\bar{x} - \lambda_2\hat{\sigma}\,\bar{y}\,\bar{y}^1 + \lambda_2 c\bar{y} - \lambda_2\hat{c}\,\bar{x}\,\bar{y}^1$$
$$+ \lambda_2\bar{x}\,\bar{y} - \hat{\sigma}\psi\,y^1 - \lambda_1\hat{\sigma}\,\bar{y}^1 + \lambda_2\bar{y}^1\,\hat{\sigma}\psi x$$
$$+ \lambda_2\hat{\sigma}\,\bar{y}\,\bar{y}^1 + \lambda_2\hat{c}\,\bar{x}\,\bar{y}^1 = 0 ---------------------------- 3.165$$

Further simplification yields

$$\hat{\sigma} - \bar{y} - \lambda_1\hat{\sigma} - \lambda_1\hat{c} + \lambda_2\hat{\sigma}\psi\,\bar{y} - \lambda_2\hat{\sigma}\,\bar{x} + \lambda_2\hat{c}\,\bar{y} + \lambda_2\bar{x}\,\bar{y} = 0 -------- 3.166$$

Introduce into equation 3.166, the expressions for the coordinate transformation of x and y from equation 3.109 and 3.110, we get.

$$\hat{\sigma} - r\sin\theta - \frac{\lambda_1}{\lambda_2} - \lambda_1\hat{\sigma}\psi - \lambda_1\hat{c} + \lambda_2\hat{\sigma}\psi\ r\sin\theta$$
$$\lambda_1\hat{\sigma}\psi + \lambda_2 r\,C\sin\theta + c\lambda_1 - \lambda_2\sigma r\cos\theta$$
$$-\hat{\sigma} + \lambda_2\left(r\sin\theta + \frac{\lambda_1}{\lambda_2}\right)\left(r\cos\theta + \frac{1}{\lambda_2}\right) ----------------- 3.167$$

Simplifying

$$r\sin\theta - \frac{\lambda_1}{\lambda_2} - r\lambda_2\sigma\psi\sin\theta + \lambda_2 cr\sin\theta - \lambda_2\sigma r\cos\theta$$
$$+ r\sin\theta + \lambda_2 r^2\sin\theta\cos\theta + r\lambda_1\cos\theta + \frac{\lambda_1}{\lambda_2} = 0 ----------------- 3.168$$

$$\hat{\sigma}\ (\lambda_2\psi\sin\theta\,r - \lambda_2\,r\cos\theta) = -\left(\lambda_2 Cr\sin\theta + \lambda_2\,r^2\sin\theta\cos\theta + r\lambda_1\cos\theta\right) --- 3.169$$

Dividing although by $r\lambda_2\cos\theta$

$$\hat{\sigma}\left(\psi\tan\theta-1\right)=-\left(C\tan\theta+r\sin\theta+\frac{\lambda_1}{\lambda_2}\right)$$

$$\hat{\sigma}=-\frac{r\sin\theta+c\tan\theta+\dfrac{\lambda_1}{\lambda_2}}{\psi\tan\theta-1}\quad ------------------------3.170$$

$$=\frac{r\sin\theta+c\tan\theta+\dfrac{\lambda_1}{\lambda_2}}{1-\psi\tan\theta}\quad ------------------------3.171$$

$$By\ \ y=y_1=0\ \ at\ \ \theta=\theta_1\quad ------------------------3.172$$

$$This\ implies\ \ y_1=r\sin\theta_1+\frac{\lambda_1}{\lambda_2}=0\quad ------------------3.173$$

$$\frac{\lambda_1}{\lambda_2}=-r\sin\theta_1\quad -------------------------3.174$$

Thus

$$\hat{\sigma}_1=\hat{\sigma}\Big|_{\theta=\theta_1}=\frac{r\sin\theta_1+c\tan\theta_1-r\sin\theta_1}{1-\psi\tan\theta_1}=\frac{c\tan\theta_1}{1-\psi\tan\theta}---3.175$$

Determination of Integration Constant (B)

The integration constant B of equation 3.162 is determined by pursuing the fact that the critical $\hat{\sigma}(x)$ determined from the solution of the Euler equation, Equation 3.107 must also satisfy the condition of transversality (i.e. the variational boundary condition), Equation 3.107 at any point on the critical rupture surface. This realized, we therefore apply the solution of $\hat{\sigma}(x)$ equations 3.159 and 3.161 (for $\psi = 0$, and $\psi \neq 0$ respectively) to the end point (r_1, θ_1) and on comparing the result with the solution, equation 3.175, for the variational boundary condition, we can solve for B, thus:

Now, the solution for the Euler second equation is

$$\hat{\sigma}(\theta)=r_0\,A_{(\theta)}+Be^{2\psi\theta}-\frac{\hat{c}}{\psi};\ \psi\neq 0$$

$$\hat{\sigma}(\theta)=r_0\sin\theta+2\hat{c}\theta+B;\psi=0$$

$$where\ A_{(\theta)}=\frac{e^{(\theta_0-\theta)\psi}}{1+9\psi^2}\left(\sin\theta-3\psi\cos\theta\right)$$

B = integration constant

Now if $\hat{\sigma}(x)$ on equations 3.159 and 3.162 is applied to the end condition, we obtain

$$\hat{\sigma}(\theta_1) = r_0 A_{(\theta_1)} + Be^{2\theta_1\psi} - \frac{c}{\psi}; = 0 \quad ----------------3.176$$

$$\hat{\sigma}(\theta_1) = r_0 \sin \theta_1 B + 2c\theta_1; = 0 ---------------------3.177$$

If these are simultaneously compared with the $\hat{\sigma}(\theta_1)$ Equation 3.175 resulting from the solution of the transversality condition, we get as follows:

$$\sigma(\theta_1) = \frac{c \tan \theta_1}{1-\psi \tan \theta_1} -------------------------3.175$$

$$Thus\ r_0 A(\theta_1) + Be^{2\theta_1} - \frac{c}{\psi} = \frac{c \tan \theta_1}{1-\psi \tan \theta_1} -----------3.178$$

$$Be^{2\theta_1\psi} = \frac{c \tan\theta_1}{1-\psi \tan\theta_1} + \frac{c}{\psi} - r_0 A(\theta_1)$$

$$= \frac{\psi \hat{c} \tan \theta_1 + \hat{c} - \psi \hat{c} \tan\theta_1}{\psi (1-\psi \tan\theta_1)} - r_0 A_{(\theta_1)} --------------3.179$$

$$= \frac{c}{\psi (1-\psi \tan\theta_1)} - r_0 A_{(\theta_1)}$$

$$B = \frac{\hat{c}\, e^{-2\theta_1\psi}}{1-\psi \tan \theta_1} - r_0 A_{(\theta_1)} e^{-2\theta_1\psi}; \ \psi \neq 0 ----------------3.180$$

similarly for the case where ψ = 0; we have

$$r_0 \sin\theta_1 + B + 2\hat{c}\theta_1 = \frac{\hat{c}\tan\theta_1}{1-\tan\theta_1} ----------------------3.181$$

$$substituting\ \psi = 0\ leads\ us\ to$$

$$r_0 \sin\theta_1 + B + 2\hat{c}\theta_1 = c \tan\theta_1 ------------------------3.182$$

$$B = \hat{c} \tan\theta_1 - 2\hat{c}\theta_1 - r_0 \sin \theta_1 ------------------------3.183$$

$$B = \hat{c}\left(\tan\theta_1 - 2\theta_1\right) - r_0 \sin \theta_1 ------------------------3.184$$

In determining the critical normal stress distribution according to the variational solution of equation 3.159 and 3.163 for $\hat{\sigma}_0$ and supporting equations, equations 3.160 for A and 3.180 and 3.184 for B, it is necessary to define the ranges of validity of the angle θ, defining the polar coordinate system.

If we express the geometrical boundary condition of equations 3.22 and 3.23 in terms of polar coordinates by introducing equations 3.111 and 3.110 and then the results into equations 3.22 and 3.23, we obtain the following expressions:

$$\mathrm{Re}\,call,\ \bar{y}^1 = \bar{y}(\bar{x} = \bar{x}_1) = 0$$

$$\bar{y}_0 = \bar{y}(\bar{x} = \bar{x}_0) = 0$$

$$\bar{x} = \bar{r}\sin\theta + \frac{1}{\lambda_2}$$

$$\bar{y} = \bar{r}\sin\theta + \frac{\lambda_1}{\lambda_2}$$

$$\bar{x}_r = \frac{1}{\lambda_2}$$

$$\bar{y}_r = \frac{\lambda_1}{\lambda_2}$$

Introducing Equations 3.111 and 3.112 respectively into equations 3.109 and 3.110 gives

$$\bar{x} = \bar{r}\cos\theta + \bar{x}_r \text{ ---------------------- } 3.185$$

$$\bar{y} = \bar{r}\sin\theta + \bar{y}_r \text{ ---------------------- } 3.186$$

Introducing equations 3.185 and 3.186 into equations 3.22 and 3.23 gives

$$\bar{x}_0 = \bar{r}_0\cos\theta_0 + \bar{x}_r = 0 \text{ ---------------------- } 3.187$$

$$\bar{y}_0 = \bar{r}_0\sin\theta_0 + \bar{y}_r = 0 \text{ ---------------------- } 3.188$$

$$\bar{y}_1 = \bar{r}_0\sin\theta_0\exp\left[(\theta_0 - \theta_1)\psi\right] + \bar{y}_r = 0 \text{ ------- } 3.189$$

From equations 3.187 and 3.188 rearranged,

$$\bar{x}_r = -r_0\cos\theta_0 \text{ ------------------------------ } 3.190$$

$$\bar{y}_r = -r_0\sin\theta_0 \text{ ------------------------------ } 3.191$$

$$r_0\sin\theta_1\exp(-\theta_1\psi) = -y_1\exp[-\theta_0\psi] \text{ --------------- } 3.192$$

Equation 3.192 shows that the relation between θ_1 and θ_0 is of the form

$$f(\theta_0) = f(\theta_1) \text{ -------------------------- } 3.193$$

$$where\ f(\theta) = \sin\theta\exp(-\theta\psi)$$

This function is only positive is the range

$$0 < \theta < \bar{\Lambda}\, and\ \theta_0 > \theta_1 \text{ ----------------------- } 3.194$$

except θ = θ_1, $\bar{\Lambda}$, in which $f(\theta) = f(0) = f(\theta_1) = 0$

The function also reaches maximum value at $\left(\overline{\Lambda}/2 - \phi\right)$.. The range of θ_0, θ_1 are therefore:

$$0 \leq \theta_1 \leq \overline{\Lambda}/2 - \phi \text{ -------------------------- } 3.195$$

$$\overline{\Lambda}/2 - \phi < \theta_0 \leq \overline{\Lambda} \text{ -------------------------- } 3.196$$

Based on the foregoing, the following ranges of θ are calculated for various values of ϕ (internal frictional angle).

Table 1: Computed values of θ_0 *and* θ_1

ϕ^0	$\theta_0 : \overline{\Lambda}/2 - \phi \leq \theta_0 \leq \overline{\Lambda}$	$\theta_1 : 0 \leq \theta_1 \leq \overline{\Lambda}/2 - \phi$
0	$\overline{\Lambda}/2 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq \overline{\Lambda}/2$
5	$1.48 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.48$
10	$1.396 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.39$
15	$1.31 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.31$
20	$1.22 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.22$
25	$1.13 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.13$
30	$1.047 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 1.047$
35	$0.96 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 0.96$
40	$0.872 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 0.872$
45	$0.7854 \leq \theta_0 \leq \overline{\Lambda}$	$0 \leq \theta_1 \leq 0.7854$

Precise values for θ_0 and θ_1 are obtained for various ϕ values by choosing θ_0 within the range tabulated and using equation 3.192 modified by the satisfaction of equation 3.194, the corresponding θ_1 value is obtained.

It is convenient to take r_0 to equal the foundation width; i.e. r_0 = B. For a foundation of total width 2.40m, r_0 = 2.40m. It must be pointed out that the radius r_0 that defines the point (r_0, θ_0) of the polar coordinates of the point (x_0, y_0) would vary with the cohesive strength for various soils. Consequently, from equation 3.8.13,

$$r = r_0 \, e^{(\theta - \theta_0)\psi}$$

the shape of the critical surface r(θ) depends on both c and ϕ also. This result is different from that of the classical solutions in which the shape of the critical rupture surface is independent of cohesion.

CONCLUSION

A cursory equation for the computation of the critical normal stress distribution on the rupture surface from determinable strength parameters of the soil is also evolved. Since the analysis here admits the logarithmic spiral curve for the critical rupture surface, the magnitude of the normal stress distribution on the surface is rightly observed to vary with position on the curve.

This result is evidently an improvement over the normal stress equation suggested by De Beer [31] and given by $\boldsymbol{\sigma}_0 = \frac{1}{4}(q_0 - 3q)(1 - \sin\boldsymbol{\varphi})$ *in which*

$q_0 = cNc + qNq + \frac{1}{2}\gamma BN\gamma$ and $q = \gamma H$

and which limits σ_0 to a constant value.

APPENDIX A

Perform the integration $\int e^{-3\psi\theta} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta}$

The above integration is carried out by parts.

Let $u = e^{-3\psi\theta}, \ dv = \cos\theta\, d\theta$ ------------------(1)

$$\frac{du}{d\theta} = -3\psi e^{-3\psi\theta}$$

$$du = -3\psi e^{-3\psi\theta}\, d\theta \text{ -------------------(2)}$$

$v = \int dv = \int \cos\theta\, d\theta = \sin\theta$ ---------------(3)

Now $e^{-3\psi\theta} \cos\theta\, d\theta = \int u dv$ -----------------(4)

But $\int u dv = uv - \int v dv$ --------------------(5)

Substituting

$\int u dv = e^{-3\psi\theta} \sin\theta - \int(-3\psi e^{-3\psi\theta} \sin\theta\, d\theta = e^{-3\psi\theta} \sin\theta + 3\psi \int e^{-3\psi\theta} \sin\theta d\theta$ ---(6)

integrating again the cycle function

$$\int e^{-3\psi\theta} \sin\theta d\theta :$$

$$Let\ u = e^{-3\psi\theta} \quad dv = \sin\theta d\theta$$

$$du = -3\psi e^{-3\psi\theta}; \quad v = -\cos\theta$$

$$\int e^{-3\psi\theta} \sin\theta d\theta = -e^{-3\psi\theta} \cos\theta - 3\psi \int e^{-3\psi\theta} \cos\theta d\theta \text{ --------------- } (7)$$

Substitute (7) into (6), then

$$e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta} = e^{-3\boldsymbol{\psi\theta}} \sin\boldsymbol{\theta} + 3\boldsymbol{\psi} \left(-e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta} - 3\boldsymbol{\psi}\int e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta}\right) \text{ ---- } (8)$$

$$= e^{-3\boldsymbol{\psi\theta}} \sin\boldsymbol{\theta} - 3\boldsymbol{\psi} e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta} - 9\boldsymbol{\psi}^2 \int e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta}$$

$$\left(1 + 9\boldsymbol{\psi}^2\right) \int e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta} = e^{-3\boldsymbol{\psi\theta}} \sin\boldsymbol{\theta} - 3\boldsymbol{\psi} e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta} = e^{-3\boldsymbol{\psi\theta}} \left(\sin\boldsymbol{\theta} - 3\boldsymbol{\psi}\cos\boldsymbol{\theta}\right)$$

$$\int e^{-3\boldsymbol{\psi\theta}} \cos\boldsymbol{\theta}\, d\boldsymbol{\theta} = \frac{e^{-3\boldsymbol{\psi\theta}}}{1 + 9\boldsymbol{\psi}^2} \left(\sin\boldsymbol{\theta} - 3\boldsymbol{\psi}\cos\boldsymbol{\theta}\right) \text{ -------------- } (9)$$

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