PRESENTATION OF INVERSE SEMIGROUP AS GRAPH

Mogbonju, M. M.
Department of Mathematics, University of Abuja. Abuja.

Mokanjuola, S. O.
Department of Mathematics, University of Ilorin, Ilorin, Kwara State, Nigeria.

Adeniji, A.O.
Department of Mathematics, University of Abuja. Abuja.

ABSTRACT

Generally, semigroups can be shown as graphs. In this paper we present how inverse semigroups are associated with graph as G = (X, A) where X is the set of vertices or nodes of (X, A) and A is its set of edges or arrows. In particular, we give some examples of inverse semigroup associated with graphs. Thus an inverse semi group graph (X, A , •) to be graph (X, A) such that (X,•) is an inverse semi group and for every $x_1, x_2, x_3 \in X$, $(x_1, x_2) \in A$. it implies that $(x_3x_1, x_3x_2) \in A$. Inverse semigroup was epresented by the aid of graph with multiple edges. In view of the findings of this work, it can be extended by presenting other types of semigroups, nonregular simple, bisimple, pre-simigroups or other semigroups as graphs. Hence, we represent inverse semigroup by the aid of graph with multiple edges.
Keywords: Semigroup , graph, Inverse semigroup, isomorphisms, homomorphism.

INTRODUCTION

There has been much work done recently on the action of semigroup on set with some important application, for example, the theory and structure of semigroups amalgams. It seems natural to consider the actions of semigroups on set "with structure and in particular on graphs. The theory of group actions has proved a powerful tool in combinatorial group theory and it is reasonable to expect the useful techniques in semigroup theory may be obtained by trying to "port" the Bass-serve theory to a semigroup context .The possibility of using graphs effectually in semigroup theory, repeating the successes of group theory open up tantalizing vistas .The work of Munn (1976) on free inverse semigroups on which he used graphs effectually to determine canonical forms for the elements of a free universe semigroup shows how powerful and illuminating graphs can be. Gardner (2006) stated the study of graphs is known as graph theory, and was first systematically investigated by D. Konig in the 1930s. Unfortunately as Gardner's (2006) notes, the confusion of this term (that is the term "graphs" to describe a network vertices and edge) with the "graph" of analytical geometry [that is, plot of functions] is regrettable. Gary and Linda (1979) define a graph G is a finite non-empty set of object called vertices (the singular as vertex) together with a (possible empty) set is unordered pairs of distinct vertices of G called edges. Howie (1995) stressed that inverse semigroup were studied first by Vanger (1952) and independently by Prester (1954). According to Howie (1995), Vanger called them "generalized group". The origin of the ideal in both cases was the study of semigroups of partial one-to-one mappings of a set, and one of the earlier results (analogous to Cayley' theorem in group theory) to the effect that every inverse semigroup has a faithful representation as an inverse semigroup of partial one-to-one mapping. According to Warren and Dunwoody (1989) an automorphism of graph *G* is an isomorphism of *G* with itself, that is, a permutation on *V*(*G*) that preserves adjacently.

PRELIMINARIES

We first recalled some important definitions needed for the development of this paper, but full details about semigroups, inverse semigroups graph (Gardner 2006, Gary and Linda 1979, Howie 1995, 1976, Lowell and Robin 1988, Munn 1974, Renshaw 2004, Yainamura 1977).

Definition (1) [3]

Let S be a non - empty set on which is defined a binary operation (*). Then the system (S, *) [i.e. the algebraic system consisting of the set S and the binary operation * on S] is called a semigroups if and only if

(i) S is closed under * and

(ii) * is associative on S

if $a, b, c \in S$ then $a * (b * c) = (a * b) * c$

Definition (2) [3]

A semigroup (S,*) is said a commutative semigroup if and only if it is commutative on S. that is $\forall$ a, b, $\in$ S, a* b = b* a.

Definition (3) [3]

Let (S, *) be a semigroup with an identity element **e** and let x $\in$ S. An element y belonging S is called an inverse of x (on the semigroup) if and only if x * y = y * x = x and y * x = x* y = y

Definition (4) [3]

A semigroup S is called a union of groups if each of its elements is contained in some subgroup of S. If x is an element of such semigroup then x € G, subgroup of S. If we denote the identity of G by **e** then within the group G we have ex = xe = x , $x^{-1}xx^{-1} = x^{-1}$

Definition (5) [4]

A semigroup S is called an inverse semigroup if every x in S possesses a unique inverse. I.e. if there exist a unique element x^{-1} in S such that $x x^{-1} x = x$, $x^{-1} x x = x^{-1}$

Theorem 1

The following statements about a semigroup S are equivalent

(a) S is an inverse semigroup

(b) S is regular and idempotent commute

Proof

To show that (a) = (b)

Let e,f, be idempotent and let $x = (ef)^{-1}$

Then $efxef = ef$, $xefx = x$

$(fxe)^2 = f(xefx)e = fxe$

Also $(ef)(fxe)(ef) = efxef = ef$

$(fxe)(ef)(fxe) = f(xefx)e = fxe$

And so ef is an inverse of fxe. But fxe, being idempotent, is it own unique inverse, and so fxe =ef.

It follows that ef is idempotent, and similarly we obtain that fe is idempotent.

$(ef)(ef)(ef) = (ef)^2 = ef$

$(fe)(ef) = (fe) = (fe)^2 = fe$

And so fe is an inverse of ef. But ef being idempotent in its own unique inverse, and so we finally obtain ef =fe.

Proposition (2.3) [3]

(a) $(a^{-1})^{-1} = a$ for every a in S

(b) $e^{-1} = e$ for every e in E (we write E (S) or simply E, for the set of idempotents of the inverse semigroups S

(c) $(ab)^{-1} = b^{-1} a^{-1}$ for every a, b,in S

(d) $a e a^{-1} \in E$, $a^{-1} e a \in E$ for every a in S and every e in E

(e) $a R b$ if and only if $aa^{-1} = bb^{-1}$, $a \alpha b$ if and only if $a^{-1}a = b^{-1}b$.

Proof

Part (a) follows by the mutuality of the inverse property, and b in immediate. To prove (c) , notice that since bb^{-1} and $a^{-1}a$ are idempotent.

$(ab)(b^{-1}a^{-1})(ab) = a(bb^{-1})(a^{-1}a)b = aa^{-1} = abb^{-1}b = ab$

$(b^{-1}a^{-1})(ab)(b^{-1}a^{-1}) = b^{-1}(a^{-1}a)(bb^{-1})a^{-1} = b^{-1}bb^{-1}a^{-1}aa^{-1} = b^{-1}a^{-1}$

Thus $b^{-1}a^{-1}$ is an inverse, and hence the inverse of ab, That is $(ab)^{-1} = b^{-1}a^{-1}$

To prove (d). Note that

$(aea^{-1})^2 = ae(a^{-1}a)ea^{-1} = aa^{-1}ae^2a^{-1} = aea^{-1}$ and similarly $(a^{-1}ea)^2 = a^{-1}ea$.

Proposition (2.4)

Let S be an inverse semigroup. Let T be a semigroup and let $\emptyset : S \rightarrow T$ be a homomorphism. Then $S\emptyset$ is an inverse semigroup.

Proof

It is immediate that $S\emptyset$ is regular. If g,h are idempotents in $S\emptyset$ then Lallelement's Lemma (Year) (Lemma 11:4,6) there exists idempotents e,f in such that $e\emptyset = g$ and $f\emptyset = h$

Hence $gh = (e\emptyset)(f\emptyset) = (ef)\emptyset = (fe)\emptyset = (f\emptyset)(e\emptyset) = hg$ and so $S\emptyset$ is an inverse semigroup.

The natural order relation on an inverse semigroup

If a,b are element of an inverse semigroup S, let us write $a \le b$ if there exists an idempotent e in S such that $a = eb$.

Lemma (2)

The relation $\le$ defined above is a partial order relation on the inverse semigroup S .

Proof

To show that $\le$ is reflexible we need only that for any a in S we have $a = ea$,where $e = a a^{-1}$

If $a = eb$ and $b = fa$, where $e,f \in E$, then $e a = e(eb) = eb = a$

And so $a = eb = ef a fea = fa = b$. Then $\le$ is anti – symmetric

If $a = et$ and $b = fe$ $(e,f \in E)$ Then $a = (ef) e$: $\le$ is transitive.

GRAPH

Definition 6

A graph is an object consisting of two set called its vertex set and its edge set the vertex set is a finite non-empty set. The edge set may be empty set; otherwise their elements are two subsets of the vertex set.

Definition 7

A graph $G = (X, A)$ consists of:

(i) A finite set $X = \{ x_1, x_2, \ldots, x_n \}$ whose elements are called nodes or vertices and

(ii) A subset A of the Cartesian product $X \times X$, the element of which are called arcs or edges.

A graph can be depicted by a diagram in which nodes or vertices are represented by points in the plane, and each arc or edge $(x_i\, x_j)$ is indicated by an arrow drawn from the point representing x_i to the point representing x_j

$G = (X, A)$ where

$X = \{x_1, x_2, x_3, x_4\}$

$A = \{ (x_1,x_2), (x_1, x_4), (x_2, x_2), (x_2, x_4), (x_3,x_2), (x_4,x_1), (x_4, x_3) \}$

In another form, a graph is an ordered pair $G = (X, A)$, where $A \subseteq (X, A)$ where $A \le X \times X$. X is a set of vertices of (X, A) and A is its set of edges or arrows of (X, A) .Thus the arrows are defected and there is at most one arrow from any vertex to any other vertex; we say that the arrow (x_1, x_2), is from the vertex x_1 to the vertex x_2 .The vertex x_1 is called the origin of (x_1, x_2) and the vertex x_2 is called its terminus.

Let $G = (X,A)$ and $H = (Y, B)$ be graphs. A partial isomorphism of $G = (X, A)$ unto $H= (Y, B)$ is a mapping $\alpha : N \to Y$, of a subset N of X into Y ,which is one- to-one and which preserves, origins and terminals of arrows, that is such that for all $(x_1, x_2) \in A\ n(N \times N)$ we have $(x_1\alpha, x_2\alpha) \in B$

Let If denote the symmetric inverse semigroup on a set F. If S and T are inverse semigroup then $S \leq T$. We mean that S is an inverse semigroup of T.

Example of Inverse Semigroup associated with graphs.

a) A semigroup $\Gamma_{(X,A)}$ of all partial automorphism of (X, A), i.e. of all partial isomorphism of (X,A) into (X,A). The semigroup $\Gamma_{(X,A)} \leq I_x$ and any inverse semigroup can be faithfully represented of some $\Gamma_{(X,A)}$

b) The semigroup $S_{(X,A)} \leq I_x$. $S_{(X,A)}$ consists of all α such that $\alpha = \rho_1 o\ \rho_2\ o\ \rho_3 o \ldots o\ p_k$.

For some $K \geq 1$, $\rho_i \in I_X$ and some $\rho_i \subseteq A \cup A^{-1}$ *for* $i = 1\ 2, \ldots, k$

We can provide a characterization of $S_{(X,A)}$. First ,we need some definition .

Definition 8

A path on the graph (X,A) is a sequence $x_o,x_1, \ldots,x_n$ such that $\{ x_{i-1} ,x_i \} \in A \cup A^{-1.}$ *for* $i =1,2,3,\ldots,n$. Such a path is said to be of length n and to be from x_o to x_n. The vertex x_i is said to be *i-th* step in this path.

Two paths, one from a to b and other from c to d are said to be parallel if either they are identical or alternatively, they are of the same lent length and for each *i*, the *i–th* step from *a* to *b* is different from *i–th* step from c to d.

Theorem 5

$S_{(X,A)}$, consists of all $\alpha \subseteq X \times X$ such that for some k, any two element $(a,b),(c,d)$ of α are such that there are parallel paths of length k from a to b and from c to d.

Proof

Let $\alpha \in S_{(X,A)}$, Then $\alpha = \rho_1\ o\ \rho_2\ o \ldots o\ \rho_k$, say , where each $\rho_i \leq I_X$ *and* $\rho_i \leq A \cup A^{-1}$. Thus, if (a,b) ,(c,d) both belong to α , there are paths $a = a_o, a_2, \ldots, a_k = b$ and $c = c_o, c_1 ,\ldots, c_k = d$ such that $(a_{i-1},a_i) \in \rho_I$ and $(c_{i-1}, c_i) \in \rho_i$ for i=1,2,...,k. Because each $\rho_i \in I_x$, if for any i; $a_i = c_i$, then $aj = cj$ for j = 0, 1,2,…, k. Thus the above path from a to b and from c to d are parallel.

Inverse semigroup graphs

An inverse semigroup graph $(G,X, \bullet)$ to be graph (X ,A) such that $(X, \bullet)$ is an inverse semigroup and is such that $\forall\ x_1, x_2, x_3, \in X, (x_1 x_2) \in A$ where $(x_3 x_1 , x_3,x_2) \in A$.

Construction: Let $(X, \bullet)$ be an inverse semigroup. Let $A \leq XxX$

Define $A^{*} = \{(x_3\, x_1), (x_1\, x_3), (x_2\, x_1), (x_3\, x_2), (x_2\, x_3), (x_1\, x_2);\ x_1, x_2, x_3, \in X\}$

Then $(X, A^{*}, \bullet)$ is an inverse semigroup graph.

All inverse semigroup graphs are clearly obtained in this way; for if $(X, A, \bullet)$ is an inverse semigroup graph, then A* = A.

As a further examples, Let $(X, A, \bullet)$, be an inverse semigroup and Let $Y \leq X$.

Define $A_y = \{(x, xy) \mid x \in X, y \in Y\}$ Then (X, A_y) is an inverse semigroup graph

INVERSE SEMIGROUP GRAPH WITH MULTIPLES ARROWS

Definition 9

A graph $G = (V(G), A(G)) = (V,A)$, is a order pair of sets, V being the set of vertices of G and A its set arrows ,together with a pairs of mappings. For a $\in A$, $0(a)$ is called the origin and $t(a)$ is called the terminus of arrows a. Together $0(a)$ and $t(a)$ are called the end point of a. We shall denote a graph simply by it set of arrows.

We introduce a set A^{-1} disjoint from A such that $a \rightarrow a^{-1}$, $a \in A$ is a bijection of A upon A^{-1}. Set $(a^{-1})^{-1} = a$, so that $a \rightarrow a^{-1}$, $a \in A$ is the reverse bijection. Define $0(a^{-1}) = t(a)$ and $t(a^{-1}) = 0(a)$ for a in A; whence the same equations hold for $a \in A^{-1}$ .The extension of the mapping 0 and t makes $(V, A \cup A^{-1})$ a graph. We denote this graph by Γ^{*} and call it the inverse closure of Γ

A path p, of length k in Γ^{*}, from α to β is a sequence $P = a_1\, a_2\, a_3 \ldots, a_k$ for which $a_i \in A \cup A^{-1}$ and $t(a) = 0(a_{i+1})$, $i = 1,2, \ldots, k-1$ and where $\alpha = 0(a_i)$, $\beta = t(a_k)$. Define $0(p) = \alpha$, the origin of the path p, and $t(p) = \beta$, the terminus of p. The path p from α to β is said to be connect α and β. Denote by p (Γ^{*}) the set of all paths on Γ^{*} than the pair $(V(\Gamma^{*}), p(\Gamma^{*}))$, together with the extensions of o and t $p(\Gamma^{*})$ just defined is a graph .We shall usually denote this graph simply by $p(\Gamma^{*})$

The relation in vertices of Γ(or Γ^{*}) such α and β are in that relation if and only if $\alpha = \beta$ or α and β are in connected by a path in Γ^{*} is an equivalence relation on $V(\Gamma)$.The equivalence classes of this relation from the sets of vertices of the connected component of Γ(and of Γ^{*}). Specifically, if V_i, $i \in I$, are the equivalence classes set $A_i = \{a \in A \mid 0(a) \in V_i, t(a) = V_i\}$ Then $\Gamma_i = (V_i, A_i)$, $i \in I$, are the connected components of Γ and $\Gamma i^{*} = V_i, A_i \cup A_i^{-1}$ are the connected components of Γ^{*}. In each component Γ_i (or Γ_i^{*}) any vertices are connected by a path in Γ^{*}.

If $p = a_i, a_2 \ldots ak$ in a path in Γ^{*} then define p^{-1} to be the path $a^{-i}\, k \ldots a_2^{-1}\, a_1^{-1}$.Thus $(p^{-1})^{-1} = p$ and $o(p) = t(p^{-1})$, $t(p) = o(p^{-1})$.

THE INVERSE SEMIGROUP GRAPH

Let p = $p(\Gamma^*)$ = be the set of paths in Γ^* and let $p'(\Gamma^*) = p' = p \cup \{o\}$, where $o \notin p$, Define a product on p' by:

$$UV = \begin{cases} \text{the path uv, if u, v} \in \text{p and t (u) = o (v)} \\ 0, \text{otherwise.} \end{cases}$$

Then p' is a semigroup with zero called involuntary semigroup of the graph Γ. If we set o^{-1} = o, then it may be checked that the mapping $p \rightarrow p'$, $p \in p'$; is an involution i.e. $(p^{-1})^{-1} = p$ and $(p\ q)^{-1} = q^{-1}\ p^{-1}$, for all p,q in p'. If we also set o(0) = t(0) =∅ ,adding extra vertex ∅ to $V(\Gamma)$, to give $V' = V'(\Gamma) = V \cup \{\emptyset\}$, then $P'(\Gamma^*)$ because a graph in which the graph $p(\Gamma^*)$ is embedded and with vertices $V(p') = V'$

Define the relation σ on p' by

$$\sigma = \{(y\,y^{-1}y\,,y)|y \in p'\} \cup \{y\,y^{-1}z\,z^{-1}zz^{-1}\,y\,y^{-1})|\,y, z \in p'\}$$

Let σ^*be the congruence in p' generated by σ

Lemma (5.1) (Warren and Dunwoody, 1989): p^o / σ^* is an inverse semigroup, indeed it is maximal inverse semigroup morphic image of p^o

Definition (10):

$p'(r) / \sigma^*$ will be denote by $k(r)$ and called the inverse semigroup of the graph Γ

Lemma (5.2) (Gary and Linda, 1979). If $x \in K(\Gamma)$ and $u, v \in x$, then $o(u) = o(v)$ and $t(u) = t(v)$.

Proof

If $(p,q) \in \sigma$ then it is easily checked that $o(p) = o(q)$ and $t(p) = t(q)$. Hence each σ -transition of u preserves the origin and terminus of u.

CONCLUDING REMARK

This work dealt extensively with basic semigroups especially inverse semigroup and some examples of inverse semigroup associated with graphs .Finally we presented how various semigroups are associated with a graph- stating with semigroups of paths ,then the inverse semigroup generated by the arrows of the graph and showed how the inverse semigroup can be represented as a graph . In view of this finding, the work can be extended by presenting other types of semigroup; nonregular simple, bisimple, free semigroups or other semigroup as graphs.

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