INTEGRAL OPERATOR PRODUCT BASIC SETS OF POLYNOMIALS IN $\mathbb{C}^n$ AND THEIR EFFECTIVENESS IN CLOSED HYPERELLIPTIC DOMAINS

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ABSTRACT

This study aimed at investigating the effectiveness, in closed hyperelliptic domains, of the set of polynomials $\{\overline{P}_m[z]\}_{m\geq 0}$ *generated by the integral product operator on basic sets. We generate, from a given basic set of polynomials in several complex variables* $\{P_m[z]\}_{m\geq 0}$, *new basic sets of polynomial* $\{\overline{P}_m[z]\}_{m\geq 0}$ *generated by the application of the integral operator* $\wedge^n$ *to the set* $\{P_m[z]\}_{m\geq 0}$. *The connection between the old set and the new set is established. All relevant properties relating to the effectiveness in closed hyperelliptic domains are properly deduced.*

INTRODUCTION

Recently, there has been an upsurge of interest in the investigation of the basic sets of polynomials (Ahmed and Kishka, 2003; Falgas, 1964; Levinson, 1944; Anjorin and Hounkonnou, 2005; Canon, 1937). The inspiration has been the need to understand the common properties satisfied by these polynomials, crucial to gaining insights into the theory of polynomials. For instance, in numerical analysis, the knowledge of basic sets of polynomials gives information about the region of convergence of the series of these polynomials in a given domain. Namely, for a particular differential equation admitting a polynomial solution, one can deduce the range of convergence of the polynomials set. This is an advantage in numerical analysis which can be exploited to reduce the computational time. Besides, if the basic set of polynomials satisfies the Cannon condition, then their fast convergence is guaranteed. The problem of derived and integrated sum sets of basic sets of polynomials in several variables has been recently treated by Ahmed and Kishka (2003) who observed that complex variables in complete Reinhardt domains and hyperelliptical regions were considered for effectiveness of the basic set.

In this study, it was aimed at investigating the effectiveness, in closed hyperelliptic domains, of the set of polynomials generated by the integral product operator on basic sets. Let us first examine here some basic definitions and properties of basic sets, useful in the sequel (Ahmed and Kishka, 2003).

Definition 1: Let $\mathbf{z} = (z_1, z_2 \ldots, z_n)$ *be an element of the space of several complex variables* $\mathbb{C}^n$*. The hyperelliptic region of radii* $r_s > 0$*,* $s \in I = \{1, 2, \ldots, n\}$*, is denoted by* $E_{[r]}$ *and its closure by* $\bar{E}_{[r]}$ *where*

$$E_{[r]} = \{w \in \mathbb{C}^n : |w| < 1\},\ \bar{E}_{[r]} = \{w \in \mathbb{C}^n : |w| \leq 1\}$$

and

$$w = \{w_1, w_2, \ldots w_n\},\ w_s = \frac{z_s}{r_s}; s \in I.$$

The unspecified domain D($\bar{E}_{[r]}$) is considered for hyperelliptic domain. Thus, the function F(z) of the complex variables z_s, $s \in I$, which is regular in $\bar{E}_{[r]}$ can be represented by the power series

$$F(\mathbf{z}) = \sum_{m=0}^{\infty} a_m z^m = \sum_{(m_1, m_2, \ldots, m_n)}^{\infty} a_{(m_1, m_2, \ldots, m_n)} (z_1^{m1} . z_2^{m^2} \ldots . z_n^{m_n}), \qquad \text{----------------(1)}$$

where $m = (m_1, m_2, \ldots m_n)$ represents the multi indices of non-negative integers for the function F(z). According to Ahmed and Kishka (2003), we have:

$$M\,[F(z), [r]] = M(F(\mathbf{z}) : (r_1, r_2, \ldots, r_n)) = \max_{\bar{E}_{[r]}} |F(z)| \qquad \text{----------------} \quad (2)$$

where $r = (r_1, r_2, \ldots, r_n)$ is the radius of the considered domain. Then for hyperelliptic domains $\bar{E}_{[r]}$ (Ahmed, 2003).

$$\sigma_m = \inf_{|t|=1} \frac{1}{t^m} = \frac{\{<m>\}^{<m>/2}}{\prod_{s=1}^{n} m_s^{m_s/2}}$$

t being the radius of convergence in the domain; $<m> = m_1 + m_2 + \ldots + m_n$ assuming $1 \leq \sigma \leq (\sqrt{n})^{<m>}$ and $m_s^{m_s/2} = 1$, whenever $m_s = 0$; $s \in I$. Since $\omega_s = \frac{z_s}{r_s}; s \in I$ $s \in$ I, we have

$$\overline{\lim}_{<m> \to \infty} \left[\frac{|a_m|}{\sigma_m \prod_{s=1}^{n} [r_s]^{<m> - m_s}} \right]^{\frac{1}{<m>}}$$

$$= \overline{\lim}_{<m> \to \infty} \left\{ \frac{\left| a_m^{\frac{1}{<M>}} \right|}{\sigma_m^{\frac{1}{<m>}} [\prod_{s=1}^{n} [r_s]]^{\frac{<m> - m_s}{<m>}}} \right\} \leq \frac{1}{\prod_{s=1}^{n} r_s}$$

Definition 2: *A set of polynomials* $\{P_m(\mathbf{z})\}_{m \geq 0} = \{P_0(z), P_1(z), \ldots\}$ *is said to be basic when every polynomial in the complex variables* z_s*;* $s \in I$ *can be uniquely expressed as a infinite linear combination of the elements of the basic set* $\{P_m(\mathbf{z})\}_{m \geq 0}$*.* Thus, according to Ahmed and Kishka (2003), the set $\{P_m(\mathbf{z})\}_{m\geq 0}$ will be basic if and only if there exists a unique row-finite-matrix $\overline{P}$ such that $\overline{P}P = P\overline{P} = \prod$, where $P = [P_{m,h}]$ is a matrix of coefficients of the set $\{P_m[z]\}_{m\geq 0}$; $h = (h_1, h_2, \ldots, h_n)$ are multi indices of non-negative integers, P is the matrix of operators deduced from the associated set of the set $\{P_m[z]\}_{m\geq 0}$ and $\prod$ is the infinite unit matrix of the basic set $\{P_m[z]\}_{m\geq 0}$, the inverse of which is $\{\overline{P}_m[z]\}_{m\geq 0}$. We have

$$P_m[z] = \sum_{h=0}^{m} P_{m,h} z^h, z^m = \sum_{h=0}^{m} \overline{P}_{m,h} P_h[z] = \sum_{h=0}^{m} P_{m,h}\overline{P}_h[z]$$
$$\overline{P}_m[z] = \sum_{h=0}^{m} \overline{P}_{m,h} z^h. \qquad \text{---------------- (3)}$$

Thus, for the function $F(\mathbf{z})$ given in (1), we get $F(\mathbf{z}) = \sum_{m=0}^{\infty} \pi_{m,h} P_m[z]$ where

$$\pi_{m.h} = \sum_{h=0}^{m} \overline{P}_{m,h} a_h = \sum_{h=0}^{m} \overline{P}_{m,h} \frac{f^{(h)}(0)}{h_s!}, h! = h(h-1)(h-2)\ldots 3.2.1.$$

The series $\sum_{m=0}^{\infty} \pi_{m,h} P_{m,h}[z]$ is an associated basic series of F($\mathbf{z}$). Let $N_m = m_1, m_2, m_n$ be the number of non zero coefficients $\overline{P}_{m,h}$ in the representation (3).

Definition 3: *A basic set satisfying the condition*

$$\overline{\lim}_{<m>\to\infty} N_m^{\frac{1}{<m>}} = 1 \qquad \text{---------------- (4)}$$

is called a Cannon basic set. If

$$\overline{\lim}_{<m>\to\infty} N_m^{\frac{1}{<m>}} = a > 1,$$

then the set is called a general basic set.

Now, let $D_m = D_{m_1, m_2, m_3, \cdots m_n}$ be the degree of polynomials of the highest degree in the representation (3). That is to say $D_h = D_{h_1, h_2, h_3, \ldots h_n}$ is the degree of the polynomial $P_h[z]$; then $D_n \leq D_m \forall n_s \leq m_s : s \in I$ and since the element of basic set are linearly independent (Ahmed and Kishka, 2003), then $N_m \leq 1+2+\ldots+(D_m+1) < \lambda D_m$, where λ is a constant. Therefore the condition (4) for a basic set to be a Cannon set implies the following condition (Ahmed and Kishka, 2003)

$$\overline{\lim}_{<m>\to\infty} D_m^{\frac{1}{<m>}} = 1 \qquad \text{---------------- (5)}$$

For any function $F(\mathbf{z})$ of several complex variables there is formally an associated basic series $\sum_{h=0}^{\infty} P_h[z]$. When the associated basic series converges uniformly to F(z) in some domain, in other words as in classical terminology of Whittaker (2003) the basic set of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which they are represented. To study the convergence property of such basic sets of polynomials in complete hyperelliptic regions, we consider the following notation for Cannon sum

$$\sigma_m \prod_{s=1}^{n} [r_s]^{<m>} \sum_{h=0}^{m} \left|\overline{P}_{m,h}\right| M(P_h[z],[r]) = \Omega(P_m[z],[r]) \qquad \text{---------------- (6)}$$

for hyperelliptic regions [1].

BASIC SETS OF POLYNOMIALS IN SEVERAL COMPLEX VARIABLES GENERATED BY INTEGRAL OPERATOR

Now, we define the integral operator Λ^n acting on the monomial $\boldsymbol{z^m}$ as

Definition 4: Let Λ^n act on $\boldsymbol{z^m}$ as follows

$$\Lambda^n z^m = \prod_{s=1}^{k} \frac{1}{z_s^{n_s}} \int_0^{z_s} \int_0^{z_s} \int_0^{z_s} \ldots \int_0^{z_s} z^m dz_s dz_s \ldots dz_s$$

$$= \begin{cases} \prod_{s=1}^{k} \left(\frac{1}{\prod_{j=1}^{n_s} (m_s + j)} \right) z^m ; m \neq 0, \\ 1, \qquad m = 0 \end{cases} \qquad \text{---------------- (7)}$$

where we have integrated n_s-times. Then applying Λ^n on $\{P_m(\boldsymbol{z})\}_{m\geq 0}$ we obtain a polynomial set $\{ P_m^*(z) \}_{m\geq 0}$ defined by

$$P_m^*(z) = \Lambda^n P_m(z) = \sum_m \alpha_{m,h} P_{m,h} z^h ,$$

where

$$\sigma_{m,h} = \begin{cases} \prod_{s=1}^{k} \left(\frac{1}{\prod_{j=1}^{n_s} (m_s + j)} \right) z^m ; m \neq 0, \\ 1, \qquad m = 0 \end{cases}$$

The set $\{P_m^*(z)\}_{m\geq 0}$ is called the integral operator product set of polynomials of several complex variables. It is then natural to ask the question that if the parent set $\{P_m(z)\}_{m\geq 0}$ is basic would $\{P_m^*(z)\}_{m\geq 0}$ be also basic? To do this, we investigate the basic property of the set $\{P_m^*(z)\}_{m\geq 0}$. We write

$$\Lambda^n P_m(z) = \sum_m \alpha_{m,h} P_{m,h} z^h = \sum P^*_{m,h} \; z^h \qquad \text{---------------- (8)}$$

The matrix coefficient $P*$ of this set is $P* = (\alpha_{m,h}P*_{m,h})$. Also, the matrix of operators $P*$ follows from the representation $z^m = \sum_h P_{m,h} P_h[z]$ so that

$$P* = (\frac{1}{\alpha_{m,h}} P_{m,h}) \quad \text{---------------- (9)}$$

leading to $P* \overline{P}* = \prod$, where $\prod$ is the infinite unit matrix. Also, by $z^m = \sum_h P_{m,h} P_h[z]$ and

Definition 5, we have

$$\overline{P}* P* = \frac{\alpha_{m,k}}{\alpha_{m,h}} \delta^k_{h,}$$

where $\delta^k_{h,}$ is the Kronneker symbol. Hence the basic property of the integral operator product set $\{P^*_m(z)\}_{m\geq 0}$ is well defined from the parent set.

EFFECTIVENESS OF THE INTEGRAL OPERATOR PRODUCT SET OF POLYNOMIALS IN CLOSED HYPERELLIPTIC DOMAIN

Let $\{P_m(z)\}_{m\geq 0}$ be basic set of polynomials of several complex variables and $\{P^*_m(z)\}_{m\geq 0}$, the integral operator product set. We pose another question here. If the set $\{P_m(z)\}_{m\geq 0}$ is effective in the closed hyperellipse $\overline{E}_{[r]}$, is the set $\{P^*_m(z)\}_{m\geq 0}$ effective in the closed domain? The major work here is to determine the connectedness between the $M(P_m, \overline{E}_{[r]})$ and $M(P^*_m, \overline{E}_{[r]})$ of the two sets in relation to the Cannon sum of the integrated product set. We proceed as follows:

$$\Omega(P^*_m[z], \overline{E}_{[r]}) = \sigma_m \prod_{s=1}^{k} [r_s]^{<m>-m_s} \sum_{h=0}^{m} \left|\overline{P}^*_{m,h}\right| M(P^*_h[z], \overline{E}_{[r]})$$

$$= \frac{\sigma_m}{\alpha_{m,h}} \prod_{s=1}^{k} [r_s]^{<m>-m_s} \sum_{h=0}^{m} \left|\overline{P}^*_{m,h}\right| M(P^*_h[z], \overline{E}_{[r]}), \quad \text{.................(10)}$$

where $\overline{P}* = (\frac{1}{\alpha_{m,h}} P_{m,h})$ and $M(P^*_h[z], \overline{E}_{[r]}) = \max_{\overline{E}[r]} \left|P^*_h[z]\right|$. Assuming D_m be the degree of the polynomial of the highest degree in the representation (3), then by Cauchy inequality, we have

$$M(P^*_m[z], \overline{E}_{[r]} \leq \sum_{h=0}^{m} \left|\overline{P}^*_{m,h}\right| \left|z^h\right|. \quad \text{---------------- (11)}$$

But $z_h = \left|\{\prod_{s=1} z_s^{m_s}\}^h\right|, \left|\omega_s\right| < 1$ where $\omega_s = \frac{z_s}{r_s}$ implying that $\left|z_s\right| < \left|r_s\right|$.

We have

$$M(P_m^*[z],\overline{E}_{[r]} \le \sum_{h=0}^{m}\left|\overline{P}_{m,h}^*\right|\frac{\left|\prod_{s=1}^{k} r_s\right|^h}{\sigma_h} \qquad \text{---------------- (12)}$$

Since $\left|\overline{P}_{m,h}^*\right| \le \left|\frac{1}{\alpha_{m,h}}\right|\left|P_{m,h}\right|$, we arrive at the equation

$$M(P_m^*[z],\overline{E}_{[r]} \le M(P_m[z],\overline{E}_{[r]})\sum_{h=0}^{m}\left|\frac{1}{\alpha_{m,h}}\right| \le K\ NmD_m^n M(P_m[z],\overline{E}_{[r]})$$

$$\le KNmD_m^{n+2}M(P_m[z],\overline{E}_{[r]}), \qquad \text{---------------- (13)}$$

where K is a constant, (n an integer because we have integrated n_s-times) and D_m is the degree of the polynomial of the highest degree in the representation (3). Substituting (13) in (10), it becomes

$$\begin{aligned}\Omega(P_m^*[z],\overline{E}_{[r]}) &\le \sigma_m \prod_{s=1}^{k}[r_s]^{<m>-m_s}\sum_{h=0}^{m}\left|\overline{P}_{m,h}^*\right| M(P_h^*[z],\overline{E}_{[r]}) \\ &= \frac{\sigma_m}{\alpha_{m,h}} KN_m D_m^{n+2}\prod_{s=1}^{k}[r_s]^{<m>-m_s}\sum_{h=0}^{m}\left|\overline{P}_{m,h}\right| M(P_m[z],\overline{E}_{[r]}) \qquad \text{------------- (14)} \\ &= \frac{\sigma_m}{\alpha_{m,h}} KN_m D_m^{n+2}\Omega(P_m[z],\overline{E}_{[r]}).\end{aligned}$$

The Cannon function

$$\begin{aligned}\Omega(P^*,\overline{E}_{[r]}) &\le \limsup_{\langle m\rangle\to\infty}\{\frac{\sigma_m}{\alpha_{m,h}} KN_m D_m^{n+2}\Omega(P_m[z],\overline{E}_{[r]})\}^{\frac{1}{\langle m\rangle}} \\ &= \limsup_{\langle m\rangle\to\infty}\{\Omega(P_m[z],\overline{E}_{[r]})\}^{\frac{1}{\langle m\rangle}}\end{aligned}$$

$$=\Omega(P^*,\overline{E}_{[r]}) = \prod_{s=1}^{k} r_s. \qquad \text{---------------- (15)}$$

But the Cannon function is non-negative, i.e

$$=\Omega(P_m[z],\overline{E}_{[r]}) \ge \prod_{s=1}^{k} r_s, \text{ leading to } =\Omega(P_m[z],\overline{E}_{[r]}) = \prod_{s=1}^{k} r_s, \qquad \text{---------- (16)}$$

Hence, the effectiveness of the original set $\{P_m(z)\}_{m\ge 0}$ implies the effectiveness of the integrated set $\{P_m^*(z)\}_{m\ge 0}$. Thus we have prove the following result

Theorem 1: *If the Cannon basic set of polynomials* $\{P_m(z)\}_{m\geq 0}$ *in several complex variables* z_s, $s \in I$ *for which the condition is satisfied, is effective in the closed hyperellipse* $\overline{E}_{[r]}$ *, then the integral operator product set* $\{P_m^*(z)\}_{m\geq 0}$ *of polynomials associated with the set* $\{P_m(z)\}_{m\geq 0}$ *will be also effective in* $\overline{E}_{[r]}$.

CONCLUDING REMARK

This study aimed at investigating the effectiveness, in closed hyperelliptic domains, of the set of polynomials generated by the integral product operator $\wedge^n$ on basic sets. Effectiveness of the new and old set was achieved by the establishment of the relations between the coefficients of these two sets. The effectiveness here is faster compared to the one deduced by Ahmed and Kishka (2003) in integrated sum sets of basic sets of polynomials in several variables. Since product is larger than the sum the effectiveness is expected to be faster.

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