Numerical Solution of Integral Equations of Second Kind Based on Sinc Collocation Method with a Variable Transformation

John, E. D.
Department of General Studies
Akwa Ibom State Polytechnic, Ikot Osurua, Ikot Ekpene, Nigeria
E-mail: ehhnorjohn@yahoo.com

ABSTRACT

A presentation of the sinc method to Volterra-Fredholm integral equations of the second kind is being considered in this paper. A single exponential transformation that relates the real line ”! and a finite arc G is used in conjunction with the sinc method to convert the Volterra-Fredholm integral equations of the second kind to algebraic equations. The deviation of the approximate solution obtained from the exact solution is measured in terms of the maximum absolute error between them at sinc points. The exceptional accuracy of the method is illustrated with numerical examples.

Keywords: *Sinc function, Collocation method, Volterra-Fredholm integral equations*

INTRODUCTION

Numerical approximation of solutions based on sinc method is among recent tools used in the estimation of solutions of differential and integral equations. This approach has enjoyed valuable contribution from several authors as can be seen in Carlson *et al* (1997), Rashidinia and Zerebnia (2008), Mohammad *et al* (2005). Usually incorporated with the sinc method is the variable transformation that relates ”! with a finite arc G. In this paper, we extend the existing procedures based on sinc methods to integral equations of the second kind, in the form of

$$u(x) = \int_a^x k_1(x,t)u(t)dt + \int_a^b k_2(x,t)u(t)dt + g(x), a \leq x \geq b \qquad (1)$$

which is called Volterra-Fredholm integral equations, Pachpatte (2008). Here k_1, k_2 and g are given smooth functions and u is the solution to be determined. This type of equations arise from parabolic boundary value problems from the mathematical modelling of the spatio-temporal development of an epidemic and other physical and biological models, Wazwaz (2011). The existence of the solution for the equations of this type has been given by Andras (2003) on the basis of fibre-Picard theorem. We derive the collocation formula for (1) using sinc method based on the variable transform which is then applied to numerical example to test its efficiency. For this purpose, we shall give the following definitions;

ISSN: 2276-8815

Basic Definitions and Theorems

The finite term sinc series is expressed as

$$f(t) = \sum_{j=-N}^{N} f(jh)\, S(j,h)(t),\ t \in R, \qquad 2$$

where the basis function S(j,h)(t) called the sinc function is defined as

$$S(j,h)(t) = \frac{\sin \pi\left(\frac{t}{h} - j\right)}{\pi\left(\frac{t}{h} - j\right)} \qquad (3)$$

and the step size h depends on a positive integer N.

Definition 2.1

Let D be a bounded and simply connected domain, then $H^{\infty}(D)$ denote the family of functions $f \in Hol(D)$ such that is finite

where

$$\|f\|_{H^{\infty}(D)} = \sup_{Z \in D} |f(z)| \qquad (4)$$

$$\|\cdot\|_{H^{\infty}(D)} \quad |f(z)| \le K|Q(z)|^{\alpha} \qquad (6)$$

Definition 2.2

Let D be a bounded and simply connected domain, then $HC(D)$ denote the family of all functions, $f \in Lip_{\alpha}(\bar{D}) \cap Hol(D)$ such that

$$\|\cdot\|_{HC(D)} = \max_{Z \in D} f(z) \qquad (5)$$

Using the above definitions 2.2 and 2.3, we give descriptions of two important sinc spaces of approximations.

Definition 2.3

Let $a > 0$ be a constant, then $L_{\alpha}(D)$ denote family of functions $f \in H^{\infty}(D)$ for which there exists a constant K such that for all $z \in D$

where the function Q is defined by $Q(z) = (z-a)(b-z)$

The variable transform

$$x = \sigma(x) = \frac{a + b\exp x}{1 + \exp x},\ x \in R, \qquad (7)$$

with its inverse

ISSN: 2276-8815

$$t = \sigma^{-1}(x) = \log\left(\frac{x-a}{b-x}\right).$$

The mapping is such that and with the sinc points

$$x_j = \sigma(jh), j = -N, \ldots, N. \quad (8)$$

Given a constant $d > 0$, we define $D_d = \{z \in C : |im\ z| < d\}$ to denote a strip region of width 2*d*, when incorporated with (4), this definition should be considered on the translated domain

$$\sigma(D_d) = \left\{ z \in C : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \right\} \quad (9)$$

Corresponding to sinc function (3) and variable transform (7), we state in the following theorem a convergence result similar to that given by Okayama *et al* (2010).

Theorem 2.1

Let $f \in L_\alpha(\sigma(D_d))$ with $0 < d < \pi$ let *N* be a positive integer, and *h* be selected by the formula $\sqrt{\frac{\pi d}{\alpha N}}$ (10)

Then there exists a constant C which is independent of , such that

$$\max_{t \in [a,b]} \left| f(t) - \sum_{j=-N}^{N} f(\sigma(jh)S(j,h))(\sigma^{-1}(x)) \right| \le C\sqrt{N}\ \exp\left(-\sqrt{\pi d \alpha N}\right) \quad (11)$$

Definition 2.4

Let $\alpha \in (0,1)$ and D is such that $(a,b) \subset D$, then $M_\alpha(D)$ denotes the class of functions $f \in HC(D)$ which have finite limits at endpoints $f(a)$ and $f(b)$ of (a,b) such that if $\rho(x) = \exp(\sigma^{-1}(x))$

$$f(x) - f(a) = 0\left(|\rho(x)|^{\alpha}\right) as\ x \to a$$

$$f(x) - f(b) = 0\left(|\rho(x)|^{-\alpha}\right) as\ x \to b \quad (12)$$

Generalized Approximation and Quadrature on $\Gamma = (a,b)$

Let ρ be defined as

there exists $g = f - Lf \in L_\alpha(D)$

With *Lf* defined as

$$Lf(z) = \frac{f(a) + \rho(x) f(b)}{1 + \rho(x)} \quad (13)$$

ISSN: 2276-8815

The translated function

$$T[g](x) = g(x) - \frac{g(a) + \rho(x)g(b)}{1 + \rho(x)}, \qquad (14)$$

belongs to $L_\alpha(D)$ *if* $g \in M_\alpha(D)$. Stenger (1993) established that if $g \in M_\alpha(\sigma(D))$ then by equation (2) the sinc approximation can be applied to the function *Tg* as

$$T[g](x) = \sum_{j=-N}^{N} T[g](x)\sigma(jh)S(j,h)(\{\sigma\}^{-1}(x)) \qquad (15)$$

By (14), we can express $g(x)$ as

$$g(x) = T[g](x) + \frac{g(a) + \rho(x)g(b)}{1 + \rho(x)},$$

and by (15) the sinc approximation of $g(x)$ is

$$P_N[g](x) = \sum_{j=-N}^{N} T[g](x)\sigma(jh)S(j,h)(\{\sigma\}^{-1}(x)) + \frac{g(a) + \rho(x)g(b)}{1 + \rho(x)} \qquad (16)$$

which may be written as

$$P_N[g](x) = g(a)w_a + \sum_{j=-N}^{N} T[g](x)\sigma(jh)S(j,h)(\{\sigma\}^{-1}(x)) + g(b)w_b(x) \qquad (17)$$

where w_a and w_b are auxiliary basis functions defined by

$$w_a(x) = \frac{1}{1 + \rho(x)}, \; w_b(x) = \frac{\rho(x)}{1 + \rho(x)} \qquad (18)$$

By Theorem 2.1 the convergence theorem for (17) may be stated thus:

Theorem 3.1

Let $g \in M_\alpha(\sigma(D_d))$ with $0 < d < \pi$, let *N* be a positive integer and let *h* be selected by the formula (10). Then there exists a constant C which is independent of, such that

$$\|g - P_N g\|_{\infty[a,b]} \le C\sqrt{N}\exp\left(-\sqrt{\pi d \alpha N}\right) \qquad (19)$$

Let $u(x) \in M_\alpha(\sigma(D))$ and $u_N(x)$ be the exact and approximate solution of (1), while $u(x_i)$ and u_i are the exact and approximate solutions at a sinc point x_i respectively, then by (17)

$$u(x) = u(x_{-N-1})w_a(x) + \sum_{j=-N}^{N} u(x_j)S(j,h)(\{\sigma\}^{-1}(x)) + u(x_{N+1})w_b(x) \qquad (20)$$

will satisfy (1) at sinc points (8), since it is a linear combination of the sinc functions $S(j,h)(t)$ and the auxiliary basis functions $w_a(x)$ and $w_b(x)$. Here, the basis functions are considered to be fixed and the collocation points defined as

$$x = \begin{cases} a, & i = -N-1 \\ \sigma(ih), & i = -N \ldots N \\ b, & i = N+1 \end{cases}$$

ISSN: 2276-8815

With the collocation points defined as above, we set the approximate solution of (1) as

$$u_n(x)=u_{-N-1}w_a(x)+\sum_{j=-N}^{N}u_jS(j,h)(\{\sigma\}^{-1}(x))+u_{N+1}w_b(x) \quad (21)$$

Following Stenger (1993) and Haber (1993) and using (21) we approximate the integrals in (1) as follows;

$$\int_a^x k_1(x,t)u(t)dt$$
$$=k_v[w_a](x)u_{-N-1}+h\sum_{j=-N}^{N}k_1(x,t_j)\sigma'(jh)u_jJ(j,h)(x)+k_v[w_b](x)u_{N+1}+0he^{-\frac{\pi}{h}} \quad (22)$$

and

$$\int_a^b k_2(x,t)u(t)dt$$
$$=k_F[w_a](x)u_{-N-1}+h\sum_{j=-N}^{N}k_1(x,t_j)\sigma'(jh)u_j+k_F[w_b](x)u_{N+1}+0e^{-\frac{\pi}{h}} \quad (23)$$

Noting that $S(j,h)(\{\sigma\}^{-1})(x_k)=S(j,h)(\{\varphi\}^{-1})\sigma(kh)=S(j,h)kh=\delta_{kj}$

We have in the above

$$K_v[f](x)=h\sum_{j=-N}^{N}k_1(x,t_j)f(x_j)\sigma'(jh)J(j,h)(x), \quad (24)$$

$$k_F[f](x)=h\sum_{j=-N}^{N}k_2(x,t_j)f(x_j)\sigma'(jh) \quad (25)$$

and $\sigma'(x)$ is the derivative of defined at sinc points as

$$\sigma'(jh)=\sigma(jh)(1-\sigma(jh))$$

with $J(j,h)(x)=\left(\frac{1}{2}+\frac{1}{\pi}Si(\pi\frac{\sigma^{-1}(x)}{h}-j\pi)\right)$

and $Si(x)$ defined as

$$Si(x)=\int_0^x\frac{\sin u}{u}du \quad (26)$$

using (22) to (26) in (1), we obtain the collocation formula as

$$\{w_a(x_k)-K_v[w_a](x_k)-K_F[w_a](x_k)\}u_{-N-1}+\sum_{j=-N}^{N}\delta_{kj}-hk_1(x_k,t_j)\sigma'(jh)J(j,h)(x_k)u_j$$
$$-hk_2(x_k,t_j)\sigma'(jh)u_j+\{w_b(x_k)--K_v[w_b](x_k)-K_F[w_b](x_k)\}u_{N+1}$$
$$=g(x_k) \quad (27)$$

(29) represents a $(2N+3)*(2N+3)$ system of linear equations, which can be expressed more compactly as

$$(E_m-V_m-F_m)u_m=g_m \quad (28)$$

with $E_m=w_a(x_k)+\sum_{j=-N}^{N}S(j,h)(\{\sigma\}^{-1}(x_k))+w_b(x_k)$

$$V_m=K_v[w_a](x_k)+h\sum_{j=-N}^{N}k(x_k,t_j)\sigma'(jh)J(j,h)(x_k)+K_v[w_b](x_k) \quad (29)$$

$$F_m=K_F[w_a](x_k)+h\sum_{j=-N}^{N}k(x_k,t_j)\sigma'(jh)+K_F[w_b](x_k)$$

$$g_m=[g(a),g(x_{-N}),\ldots,g(x_N),g(b)]^T$$

$$u_m=[u_{-N-1},u_{-N},\ldots u_N,u_{N+1}]^T$$

ISSN: 2276-8815

By solving the above system of equations for and using the result in (21), we obtain the approximate solution $u_N(x)$ to (1).

NUMERICAL EXAMPLES

For these examples, we will find the approximate result and compute the maximum error in order to determine the accuracy of the method.

Example 1

In this example we consider equation (1) with

$$k_1(x,t) = x - t,\ k_2(x,t) = x, g(x) = -2 - 2x + 2e^x, 0 \le x \le 1$$

This problem was considered by Wazwaz (2011), using method based on series solution.

The exact solution is given as $u(x) = xe^x$

The results for Example 1 are illustrated on Table 1, Figure 1 and Figure 2 respectively.

Example 2

In this example we consider equation (1) with

$$k_1(x,t) = x - t, k_2(x,t) = x + t, g(x) = x^2 - \tfrac{1}{12}x^4 - \tfrac{1}{3}x - \tfrac{1}{4}, 0 \le x \le 1$$

This problem was considered by Wazwaz (2011), using method based on modified Adomian decomposition.

The exact solution is given as $u(x) = x^2$.

The results for Example 2 are illustrated on Table 2, Figure 3 and Figure 4 respectively.

The collocation formula described in section 3 is implemented with $a = 1$ and $d = \frac{\pi}{2}$

so that $h = \frac{\pi}{\sqrt{2N}}$. The computations are carried out using MATLAB®, and the maximum absolute error $\left|E_N(h(\sigma))\right|$ measures the largest error at the sinc points in the approximation for any chosen *N*. It is defined by

$$\left|E_N(h(\sigma))\right| = \max_{k=-N-1,N,\ldots N,N+!} \left|u(x_k) - u_N(x_k)\right| \qquad (30)$$

Where u and u_N are the exact and approximate solutions of (1), respectively.

The approximate solution in Figure 1 a maximum deviation of 4.3 x 10-3 from exact solution appears at x = 0.9964 and remains consistent as x increases towards 1 for N = 10. Figure 2 shows the rapid decrease in the maximum absolute error with the increasing value of N, which is in agreement with information from Table 1. Similarly, in Figure 3, we have a maximum absolute error of 7.2 x 10-6 at x = 0.9927 and also remaining consistent as x increases towards 1 for N = 10 and Figure 4 agrees with error decay pattern illustrated on table 2.

ISSN: 2276-8815

CONCLUSION

The numerical result of the examples presented in this work indicates exceptional accuracy of the method implemented. The deviations of the approximate solution from the exact solution as illustrated in Figures 1 and 3 verify our claim. Also, Figures 2 and 4 and Tables 1 and 2 justify increase accuracy of the results for increase in N number of evaluations. A theoretical convergence analysis of the method for equations of this type can give a clearer picture of the error bounds. We intend to discuss this in our subsequent work.

Table 1: Example 1

N	h	$\lvert E_{s}(h(\sigma))\rvert$
10	0.7025	4.300×10^{-3}
20	0.4967	2.6161×10^{-4}
30	0.4056	2.9463×10^{-5}
40	0.3512	4.6573×10^{-6}
50	0.3142	8.9677×10^{-7}
60	0.2868	2.0315×10^{-7}

Table 2: Example 2

N	h	$\lvert E_{s}(h(\sigma))\rvert$
10	0.7025	7.200×10^{-3}
20	0.4967	4.3727×10^{-4}
30	0.4056	4.9225×10^{-5}
40	0.3512	7.7134×10^{-6}
50	0.3142	1.4981×10^{-6}
60	0.2868	3.3936×10^{-7}

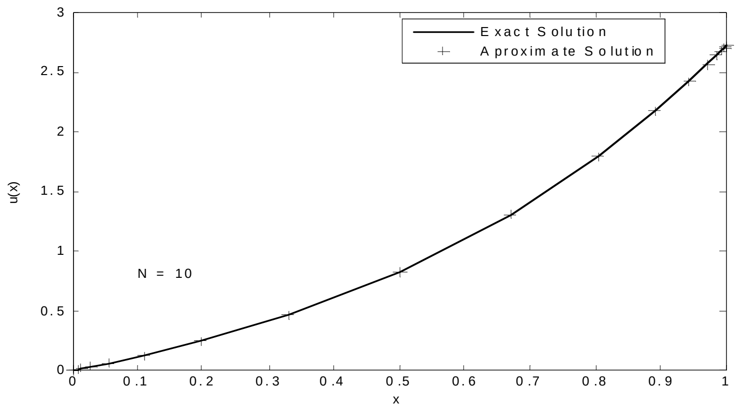

Figure 1: Exact solution and Approximate solution for Example 1

ISSN: 2276-8815

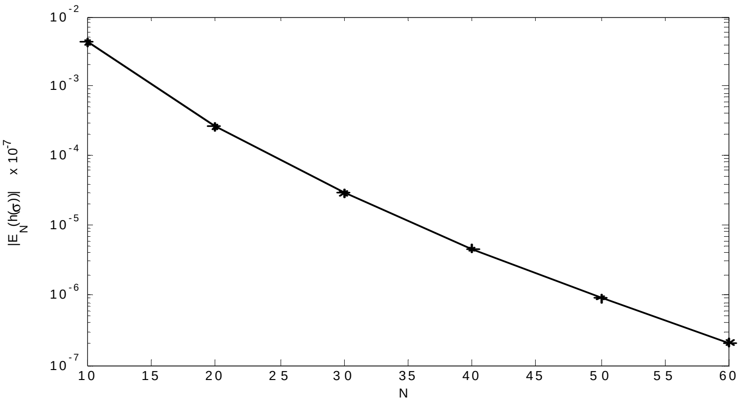

Figure 2: Maximum absolute error for Example 1

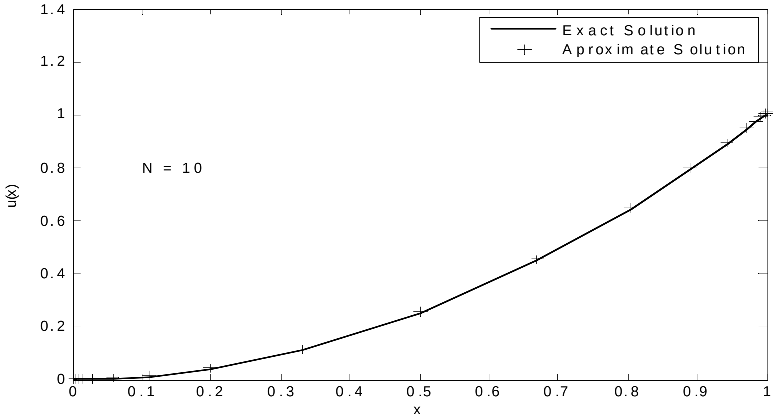

Figure 3: Exact solution and approximate solution for Example 2

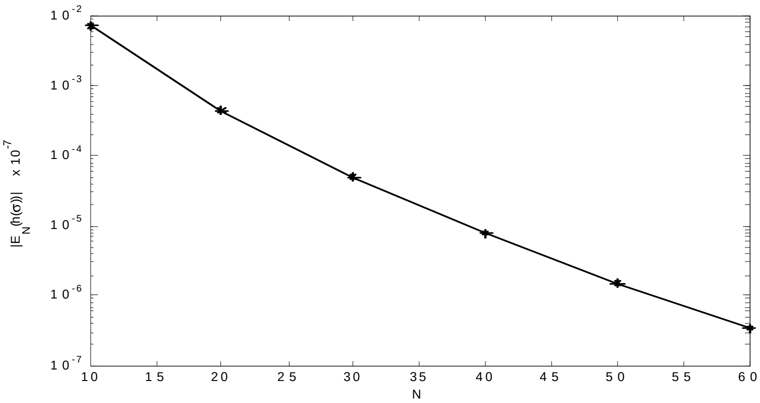

Figure 4: Maximum absolute error for Example 2

REFERENCES

Andras, S. (2003). Weakly singular Volterra and Fredholm-Volterra integral equations. Studia University *"Babes Bolyai" Mathematica,* XLVIII (3).

Carlson T. S., Dockery J. and **Lund J.** (1997). A sinc-collocation method for initial value problems. *Mathematics of Computation,* 66 (127), 215-235.

Haber, S. (1993). Two formulas for indefinite numerical integration. *Mathematics of Computation,* 60(201), 279-296.

Muhammad M., Nurumuhammad A. and **Mori M.** (2005). Numerical solution of integral equations by means of the sinc collocation method based on double exponential transformation. *Journal of Computational and Applied Mathematics,* 177, 267-286.

Okayama T., Matsuo T. and **Sugihara M.** (2011). Improvement of Sinc- collocation method for Fredholm integral equations of the second kind. *BIT Numerical Mathematics,* 51, 339-366.

Rashidinia, J. and **Zarebnia, M.** (2008). New approach for numerical solution of Volterra integral equations of the second kind. *IUST International Journal of Engineering Science,* 19(5-2), 59-65.

Pachppatte, B. G. (2008). On a Certain Volterra-Fredholm type integral equations. *Journal of Inequalities in Pure and Applied Mathematics,* 9(4), 116.

Stenger, F. (1993). *Numerical methods based on sinc and analytic functions.* New York: Springer-Verlag.

Wazwaz, Abdul-Majid (2011). *Linear and nonlinear integral equations, methods and applications.* Beijing and Springer-Verlag Heidelberg: Higher Education Press

ISSN: 2276-8815