On Close - to - Convex Function and Univalent Function

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ABSTRACT

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z)$ *be regular in the unit disk* $E = \{z : |z| < 1\}$. *In this study, a condition under which* $\operatorname{Re} \frac{f'(z)}{g'(z)} > 0$ *was establish and the result applied to discuss the univalency of the function* $f(z)$ *. AMS (MOS) Subject classification codes 30C45, 30C50.*
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INTRODUCTION

Let *A* denote the class of functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ which are regular in the unit disk $|z| < 1$. We denote the subclass of *A* consisting of regular and univalent functions $f(z)$ in the unit disk and satisfying $\operatorname{Re} \frac{f(z)}{(z)} > 0$ by S_0.

Definitions

1. A function $f(z)$ is said to be convex in the unit disk $|z| < 1$ if and only if $\operatorname{Re}\left[1 + \frac{f''(z)}{f'(z)}\right] > 0$
2. A function $f(z)$ is said to be starlike in the unit disk $|z| < 1$ if and only if $\operatorname{Re}\left[\frac{z f''(z)}{f'(z)}\right] > 0$
3. A function $f(z)$ regular in the unit disk $|z| < 1$ is said to be close-to-convex function if there is a convex function $g(z)$ such that $\operatorname{Re}\left[\frac{f''(z)}{g'(z)}\right] > 0$

It is clearly known that if $\operatorname{Re}\left[\frac{f''(z)}{g'(z)}\right] > 0$ for $|z| < 1$, and then $f(z)$ is close-to-convex. So it is natural tobe interested in the condition under which $\operatorname{Re}\left[\frac{f''(z)}{g'(z)}\right] > 0$ for $|z| < 1$ clearly, every starlike function is close-to-convex function. It is known in MacGregor (1963) that if $\operatorname{Re}\frac{f(z)}{(z)} > \frac{1}{2}$ for $|z| < 1$ then $f(z)$ is starlike for $|z| < 2^{-\frac{1}{2}}$. Many authors have worked in this direction, for example, in Marx (1932) and Strohhacker (1993). It was shown that if $f(z)$ is convex in $|z| < 1$ then $\operatorname{Re}\frac{f(z)}{(z)} > \frac{1}{2}$ (Miller and Mocanu, 1978). Also, in Strohhacker (1993) it was proved that if $f(z)$ satisfies $\operatorname{Re}(z) > \beta$ for $0 \le \beta < 1$ *and* $z \in E$ then $\operatorname{Re}\frac{f(z)}{(z)} > \frac{1+2\beta}{3}$.
This was improved in Yamagushi (1966). This result was also improved in Opoola and Fadipe-Joseph (2009), Babalola and Opoola (2007) where the condition under which $\operatorname{Re}\frac{f(z)}{(z)} > \frac{1}{2}$ was established.

Preliminary and statement of results

Let $f(z) = z + z + a_2 z^2 + a_3 z^3 + \ldots,\ |z| < 1$ and let $h(w,x) = f\left(g^{-1}(w)\right)$ be a function in the unit disk. Also let

$$g(w) = \frac{w'(z + x)\left(1 + |x|^2\right)}{f'\left(\frac{x+z}{1+\bar{x}z}\right) - f(x)}$$

It is known that $f(z)$ is univalent for $|z| < 1$ if $g(z)$ is univalent for $|z| < 1$.
Let $z = g^{-1}(w)$ *then* $g(z) = w$.
Also let $g(w) = \frac{1}{z} + h(w,x) = f^{-1}(h(w,x))$.
It is known that in Nehari (1949) that $f(z)$ is univalent for $|z| < 1$ if and if $|h'(z,x)| < 1$ for $|x| < 1\,|z| < 1$. The following lemmas shall be used to prove the main result.
Lemma [5]
If $f(z) = z + z + a_2 z^2 + a_3 z^3 + \ldots, \in S_0$ then the partial sums

$S_n(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n \ (n = 2, 3, ...)$ are univalent in $|z| < \frac{1}{4}$

Lemma [8]

If $f(z) \in S_0$ *then* $f(z)$ is univalent in $|z| < 2^{\frac{1}{2}-1}$

Lemma [8] - Wolff-Noshiro's lemma (Noshiro, 1934)

If $f(z)$ is analytic in $[z] < R$ *and* $\operatorname{Re} f'(z) > 0 \ (|z| < R)$ *then* $f(z)$ is univalent in $|z| < R$

The main result in this paper is the following theorem.

Theorem

If $f(z) = z + z + a_2 z^2 + a_3 z^3 + ...$ is regular for $|z| < 1$ and $|h(w\ x)| < 1$ for $|z| < 1, |x < 1|$ then $\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right] > \frac{1}{2}$

Proof of the main result and applications

In this section, the proof of the main result in this paper is given. Applications of the main results were also given.

Proof of the Theorem 2.4.

Let $f(z) = z + z + a_2 z^2 + a_3 z^3 + ...$ be regular for $|z| < 1.$ Consider the function

$$g(w) = \frac{w'(z+x)\left[1 + |x|^2\right]}{f'\left(\frac{x+z}{1+\bar{x}z}\right) - f(x)} = \frac{1}{z} + h(w, x)$$

Since $f(0) = 0$ and $f(0) = 1$ we have that

$$h(w,0) = \frac{w'(z)}{f'(z)} - \frac{1}{z}$$

By the condition of the theorem and since $|z| < 1$ we have that

$$\left|\frac{w'(z)}{f'(z)} - \frac{1}{z}\right| < 1$$ which implies that

$$\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right] > \frac{1}{2}$$

Since $w'(z) = g'(z)$.

As application we prove the following theorems.

Theorem

If $f(z) = z + z + a_2z^2 + a_3z^3 + \ldots$ is regular for $|z| < 1$
and $|h(w,x)| < 1$ for $|z| < 1$; $|x| < 1$,
then the partial sums
$S_t(z) = z + a_2z^2 + a_3z^3 = \ldots + a_tz^t \, (t = 2,3\ldots)$ are univalent in $|z| < \frac{1}{4}$

Proof:
By theorem (2.4) we have that

$$\mathrm{Re}\left[\frac{f'(z)}{g'(z)}\right] > \frac{1}{2}$$

Hence, by applying a theorem of Yamagushi (1966), it follows that the partial sums $S_t(z) = z + a_2z^2 + a_3z^3 = \ldots + a_tz^t \, (t = 2,3\ldots)$ are univalent in the unit disk $|z| < \frac{1}{4}$ since every starlike function is a close-to-convex function *2*

3.2 Theorem

If $f(z) = z + z + a_2z^2 + a_3z^3 + \ldots$ is regular for $|z| < 1$ and $|h(w,x)| < 1$ for $|z| < 1$; $|x| < 1$, then

$$\mathrm{Re}\ f'\left(re^{i\theta}\right) \geq \frac{1-2r-r^2}{(1+r)^2}$$

for $0 \leq r < 2\frac{1}{2} - 1$

Proof:

Let $f(z) = z + z + a_2z^2 + a_3z^3 + \ldots$ is regular for $|z| < 1$ and $|h(w,x)| < 1$ for $|z| < 1$; $|x| < 1$, then

$$\mathrm{Re}\left[\frac{f'(z)}{g'(z)}\right] > \frac{1}{2}$$

by theorem (2.4).Hence by a theorem of Yamagushi (1966), we have that

$$\mathrm{Re}\ f'\left(re^{i\theta}\right) \geq \frac{1-2r-r^2}{(1+r)^2}$$

for $0 \leq r < 2\frac{1}{2} - 1$since every starlike function is a close-to convex function. The bound is sharp

Theorem

If $f(z) = z + z + a_2 z^2 + a_3 z^3 + \dots$ is regular for $|z| < 1$ and $|h(w, x)| < 1$ for $|z| < 1$; $|x| < 1$, then $f(x)$ is univalent for $|z| < 2\frac{1}{2} - 1$

Proof:

Let $f(z) = z + z + a_2 z^2 + a_3 z^3 + \dots$ is regular for $|z| < 1$ and $|h(w, x)| < 1$ for $|z| < 1$; $|x| < 1$, then

$\mathrm{Re}\left[\frac{f'(z)}{g'(z)}\right] > \frac{1}{2}$ by theorem (2.4).Also by theorem (3.2)

$$\mathrm{Re}\ f'\left(re^{i\theta}\right) \geq \frac{1-2r-r^2}{(1+r)^2}$$

for $0 \leq r < 2\frac{1}{2} - 1$.Which implies that

$\mathrm{Re}\ f'\left(re^{i\theta}\right) > 0$

for $0 \leq r < 2\frac{1}{2} - 1$ since $g'(z) > 0$.The result follows from the well known Wolff-Noshiro's lemma (Noshiro, 1934).

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